EXERCISE SHEET 8 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
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1. (i) By the chain rule, we have $J_{f\circ G}(x,y) = J_f(G(x,y)) \cdot J_G(x,y)$ (product of matrices) $J_f(u,v,w) = \begin{pmatrix} 2u & 2v & 1 \end{pmatrix}$, and since $G : \mathbb{R}^2 \to \mathbb{R}^3$ we have $J_f(G(x,y)) = \begin{pmatrix} 2\cos x + 2\sin y & -2\sin x + 2\cos y & 1 \end{pmatrix}$,

$$J_G(x,y) = \begin{pmatrix} -\sin x & \cos y \\ -\cos x & -\sin y \\ 2\cos x\cos y & -2\sin x \sin y \end{pmatrix}.$$

Therefore, $J_{f\circ G}(x,y) = \begin{pmatrix} 2\cos x + 2\sin y & -2\sin x + 2\cos y & 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin x & \cos y \\ -\cos x & -\sin y \\ 2\cos x\cos y & -2\sin x \sin y \end{pmatrix}$
$= \begin{pmatrix} -2\sin x \sin y & 2\cos x \cos y \end{pmatrix}$.

(ii) To perform the computation directly, first see that $(f \circ G)(x,y) = 2 + 2\cos x \sin y$, such that indeed

$$J_{f\circ G}(x,y) = \begin{pmatrix} -2\sin x \sin y & 2\cos x \cos y \end{pmatrix}.$$

2.

i) We verify by a direct calculation that $x^2 + y^2 + z^2 = \rho^2$. Thus the point (x,y,z) is indeed on the sphere of radius ρ.

ii) Write $(x,y,z) = (\sqrt{6}, \sqrt{2}, -2\sqrt{2})$ in the form $(x,y,z) = G(\rho,\theta,\varphi)$. We obtain
- $\rho^2 = 6 + 2 + 8 = 16$, $\rho = 4$,
- $\cos(\theta) = (-2\sqrt{2})/\rho = -\sqrt{2}/2$, $\theta \in [0,\pi]$, $\theta = 3\pi/4$,
- $\cos(\varphi) = \sqrt{6}/(\rho\sin(\theta)) = \sqrt{6}/(4\sqrt{2}/2) = \sqrt{3}/2$,
  $\sin(\varphi) = \sqrt{2}/(\rho\sin(\theta)) = \sqrt{2}/(4\sqrt{2}/2) = 1/2$, $\varphi = \pi/6$.

We therefore have $\rho = 4$, $\theta = 3\pi/4$ and $\varphi = \pi/6$.

iii) For $(\rho,\theta,\varphi) \in \mathbb{R}_+ \times]0,\pi[\times]0,2\pi[$ the Jacobian matrix of G becomes

$$J_G(\rho,\theta,\varphi) = \begin{pmatrix} \frac{\partial G_1}{\partial \rho}(\rho,\theta,\varphi) & \frac{\partial G_1}{\partial \theta}(\rho,\theta,\varphi) & \frac{\partial G_1}{\partial \varphi}(\rho,\theta,\varphi) \\ \frac{\partial G_2}{\partial \rho}(\rho,\theta,\varphi) & \frac{\partial G_2}{\partial \theta}(\rho,\theta,\varphi) & \frac{\partial G_2}{\partial \varphi}(\rho,\theta,\varphi) \\ \frac{\partial G_3}{\partial \rho}(\rho,\theta,\varphi) & \frac{\partial G_3}{\partial \theta}(\rho,\theta,\varphi) & \frac{\partial G_3}{\partial \varphi}(\rho,\theta,\varphi) \end{pmatrix}$$

$$= \begin{pmatrix} \sin(\theta)\cos(\varphi) & \rho\cos(\theta)\cos(\varphi) & -\rho\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & \rho\cos(\theta)\sin(\varphi) & \rho\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -\rho\sin(\theta) & 0 \end{pmatrix}.$$

iv) Expanding $\det(J_G(r,\theta,\varphi))$ (see the matrix above) yields

$$\begin{aligned}\det(J_G(r,\theta,\varphi)) &= \cos(\theta)\rho^2\Big(\cos(\theta)\sin(\theta)\cos^2(\phi) + \cos(\theta)\sin(\theta)\sin^2(\phi)\Big) \\ &\quad + \rho\sin(\theta)\rho\Big(\sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi)\Big) \\ &= \rho^2\Big(\cos^2(\theta)\sin(\theta) + \sin^3(\theta)\Big) = \rho^2\sin(\theta)\,.\end{aligned}$$

v) Again, by using the matrix above, one can show that $(J_G)^\top \cdot J_G = D$, where D is the matrix diagonal whose coefficients on the diagonal are 1, ρ^2 and $\rho^2\sin^2(\theta)$ (from top to bottom), the other coefficients being zero. Hence, it follows that $(D^{-1}\cdot(J_G)^\top)\cdot J_G = I$, and thus we can simply write $(J_G)^{-1} = D^{-1}\cdot(J_G)^\top$ such that

$$(J_G(\rho,\theta,\varphi))^{-1} = \begin{pmatrix} \sin(\theta)\cos(\varphi) & \sin(\theta)\sin(\varphi) & \cos(\theta) \\ \rho^{-1}\cos(\theta)\cos(\varphi) & \rho^{-1}\cos(\theta)\sin(\varphi) & -\rho^{-1}\sin(\theta) \\ -\rho^{-1}\sin(\theta)^{-1}\sin(\varphi) & \rho^{-1}\sin(\theta)^{-1}\cos(\varphi) & 0 \end{pmatrix}.$$

3. A first restriction on D is that $x \neq -2$ and $y \neq -1/2$. To find the transformation $G = H^{-1}\colon \tilde{D} \to D$, we use

$$v = \frac{x}{2y+1} \quad \Leftrightarrow \quad x = v(2y+1) \tag{1}$$

which we substitute in the expression given for u

$$u = \frac{y}{x+2} \quad \Leftrightarrow \quad u(x+2) = y \quad \overset{(1)}{\Leftrightarrow} \quad uv(2y+1) + 2u = y$$
$$\Leftrightarrow \quad uv + 2u = y(1-2uv) \quad \Leftrightarrow \quad y = \frac{u(v+2)}{1-2uv}\,,$$

provided that $1-2uv\neq 0$. Replacing y in the equation on the right in (1), we find

$$x=\frac{2uv(v+2)}{1-2uv}+v=\frac{2uv(v+2)+v-2uv^2}{1-2uv}=\frac{(4u+1)v}{1-2uv}\ ,$$

so that the two relations

$$u=\frac{y}{x+2}\ ,\ v=\frac{x}{2y+1},$$

are equivalent to the two relations

$$x=\frac{(4u+1)v}{1-2uv}\ ,\ y=\frac{u(v+2)}{1-2uv}\ ,$$

provided that $x\neq -2$, $y\neq -1/2$ and $uv\neq 1/2$. We can therefore put

$$G(u,v)=\frac{1}{1-2uv}\Big((4u+1)v,\ u(v+2)\Big),$$

and choose

$$\begin{aligned}D&=\left\{(x,y)\in\mathbb{R}^2 : x\neq -2,\ y\neq -1/2 \ \text{ and } \ \frac{xy}{(x+2)(2y+1)}\neq 1/2\right\}\\&=\left\{(x,y)\in\mathbb{R}^2 : x\neq -2,\ y\neq -1/2 \ \text{ and } \ x+4y+2\neq 0\right\}\end{aligned}$$

And

$$\begin{aligned}\tilde{D}=H(D)&=\left\{(u,v)\in\mathbb{R}^2 : \frac{(4u+1)v}{1-2uv}\neq -2,\ \frac{u(v+2)}{1-2uv}\neq -1/2 \ \text{ and } \ uv\neq 1/2\right\}\\&=\left\{(u,v)\in\mathbb{R}^2 : v\neq -2,\ u\neq -1/4 \ \text{ and } \ uv\neq 1/2\right\}.\end{aligned}$$

H and G are indeed of class C^1 over open sets D and $\tilde{D}$ respectively. The condition $x+4y+2\neq 0$ has been added so that the calculations above are possible. To ensure that this condition is unavoidable, in other words, to see that D cannot be expanded, let's study the determinant of the Jacobian $\det(J_H)(x,y)$ if $x\neq -2$ and $y\neq -1/2$ (these two conditions are necessary for H to be defined). We obtain

$$J_H(x,y)=\begin{pmatrix}-\frac{y}{(x+2)^2} & \frac{1}{x+2}\\ \frac{1}{2y+1} & -\frac{2x}{(2y+1)^2}\end{pmatrix}$$

and

$$\det(J_H)(x,y)=\frac{2xy-(x+2)(2y+1)}{(x+2)^2(2y+1)^2}=-\frac{x+4y+2}{(x+2)^2(2y+1)^2}\ .$$

Since the Jacobian vanishes if $x + 4y + 2 = 0$, D cannot be enlarged. Indeed, on any open set D as in the statement, $J_H(x, y)$ is necessarily an invertible matrix, with inverse $J_G(H(x, y))$.

4. We have that $f(u, v) = g(ve^{-2u}, u^2e^{-v}, u)$ and then $f(1, 0) = g(0, 1, 1)$. Then applying the chain rule we have

$$\begin{aligned}
\left(\frac{\partial f}{\partial u}(1,0), \frac{\partial f}{\partial v}(1,0)\right) &= \nabla f(1,0) \\
&= \nabla g(0,1,1) \cdot \mathbf{J}_h(1,0) \\
&= \left(\frac{\partial g}{\partial x}(0,1,1), \frac{\partial g}{\partial y}(0,1,1), \frac{\partial g}{\partial z}(0,1,1)\right) \cdot \begin{pmatrix} \frac{\partial(ve^{-2u})}{\partial u}\Big|_{(1,0)} & \frac{\partial(ve^{-2u})}{\partial v}\Big|_{(1,0)} \\ \frac{\partial(u^2e^{-v})}{\partial u}\Big|_{(1,0)} & \frac{\partial(u^2e^{-v})}{\partial v}\Big|_{(1,0)} \\ \frac{\partial(u)}{\partial u}\Big|_{(1,0)} & \frac{\partial(u)}{\partial v}\Big|_{(1,0)} \end{pmatrix},
\end{aligned}$$

hence

$$\begin{aligned}
\frac{\partial f}{\partial u}\Big|_{(1,0)} &= \frac{\partial g}{\partial x}(0,1,1) \cdot \frac{\partial(ve^{-2u})}{\partial u}\Big|_{(1,0)} + \frac{\partial g}{\partial y}(0,1,1) \cdot \frac{\partial(u^2e^{-v})}{\partial u}\Big|_{(1,0)} + \frac{\partial g}{\partial z}(0,1,1) \cdot \frac{\partial(u)}{\partial u}\Big|_{(1,0)} \\
&= \frac{\partial g}{\partial x}(0,1,1) \cdot (-2ve^{-2u})\Big|_{(1,0)} + \frac{\partial g}{\partial y}(0,1,1) \cdot (2ue^{-v})\Big|_{(1,0)} + \frac{\partial g}{\partial z}(0,1,1) \cdot 1 \\
&= 2\frac{\partial g}{\partial y}(0,1,1) + \frac{\partial g}{\partial z}(0,1,1).
\end{aligned}$$

5. We have $f(u, v) = g(u + v, u - v)$ and $f(1, 1) = g(2, 0)$. Thus applying the chain rule we have

$$\begin{aligned}
\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \mathbf{J}_f(1,1) = \mathbf{J}_g(2,0) \cdot \mathbf{J}_h(1,1) \\
&= \begin{pmatrix} \frac{\partial g_1}{\partial x}(2,0) & \frac{\partial g_1}{\partial y}(2,0) \\ \frac{\partial g_2}{\partial x}(2,0) & \frac{\partial g_2}{\partial y}(2,0) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial g_1}{\partial x}(2,0) + \frac{\partial g_1}{\partial y}(2,0) & \frac{\partial g_1}{\partial x}(2,0) - \frac{\partial g_1}{\partial y}(2,0) \\ \frac{\partial g_2}{\partial x}(2,0) + \frac{\partial g_2}{\partial y}(2,0) & \frac{\partial g_2}{\partial x}(2,0) - \frac{\partial g_2}{\partial y}(2,0) \end{pmatrix}.
\end{aligned}$$

By solving the equations occurring from the above equality we obtain that

$$\frac{\partial g_1}{\partial x}(2,0) = \frac{\partial g_1}{\partial y}(2,0) = 1, \quad \frac{\partial g_2}{\partial x}(2,0) = 1, \quad \frac{\partial g_2}{\partial y}(2,0) = -1.$$

6. The electric field of a charge Q is given by

$$\mathbf{E}(x,y,z) = \frac{\varepsilon Q x}{(x^2+y^2+z^2)^{3/2}}\mathbf{e_1} + \frac{\varepsilon Q y}{(x^2+y^2+z^2)^{3/2}}\mathbf{e_2} + \frac{\varepsilon Q z}{(x^2+y^2+z^2)^{3/2}}\mathbf{e_3}.$$

Consider the function

$$f(x,y,z) = -\frac{\varepsilon Q}{\sqrt{x^2+y^2+z^2}}$$

and then we calculate

$$\frac{\partial f}{\partial x}(x,y,z) = -\varepsilon Q\left(-\frac{\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)}{x^2+y^2+z^2}\right) = \frac{\varepsilon Q x}{(x^2+y^2+z^2)^{3/2}},$$

and similarly

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\varepsilon Q y}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial z}(x,y,z) = \frac{\varepsilon Q z}{(x^2+y^2+z^2)^{3/2}}.$$

Then it follows that $\mathbf{E} = \nabla f$, hence $\mathbf{E}$ is conservative with potential function f.

7. (a) Since $\mathbf{F}$ is conservative then there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. Thus,

$$P = \frac{\partial f}{\partial x}, \qquad Q = \frac{\partial f}{\partial y}$$

and then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x}.$$

(b)

i) $\mathbf{F}(x,y) = (2x\sin(2y) - 3y^2)\mathbf{e_1} + (2 - 6xy + 3x^2\cos(2y))\mathbf{e_2}$ and we have that

$$\frac{\partial(2x\sin(2y) - 3y^2)}{\partial y} = 4x\cos(2y) - 6y, \qquad \frac{\partial(2 - 6xy + 3x^2\cos(2y))}{\partial x} = -6y + 6x\cos(2y)$$

and since these are not equal, in view of (a), $\mathbf{F}$ is not conservative.

ii) $\mathbf{G}(x,y) = (2 - 6xy + y^3)\mathbf{e_1} + (x^2 - 8y + 3xy^2)\mathbf{e_2}$ and we have that

$$\frac{\partial(2 - 6xy + y^3)}{\partial y} = -6x + 3y^2, \qquad \frac{\partial(x^2 - 8y + 3xy^2)}{\partial x} = 2x + 6y^2$$

and since these are not equal, in view of (a), $\mathbf{G}$ is not conservative.

8. We set $g_1(x,y,z) = 4x - 3y - 9,\ g_2(x,y,z) = x^2 + z^2 - 9$. We have $\nabla g_1(x,y,z) = (4,-3,0)$ and $\nabla g_2(x,y,z) = (2x,0,2z)$, which are linearly independent for any $(x,y,z) \in \mathbb{R}^3$. We apply the method of Lagrange multipliers and we start with the equation

$$\nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z),$$

which gives us the following system of equations

$$\begin{cases} 6x = 4\lambda_1 + 2x\lambda_2 \\ 1 = -3\lambda_1 \\ 0 = 2z\lambda_2 \\ 4x - 3y = 9 \\ x^2 + z^2 = 9 \end{cases}.$$

The second equation gives $\lambda_1 = -\frac{1}{3}$ and the third one gives $z = 0$ or $\lambda_2 = 0$.

- If $z = 0$, then from the third equation we get $x = \pm 3$ and then from the fourth one we get $y = 1$ (for $x = 3$) or $y = -7$ (for $x = -3$).
- If $\lambda_2 = 0$, then, using that $\lambda_1 = -\frac{1}{3}$, the first equation gives $x = -\frac{2}{9}$ and then the fourth one gives $y = -\frac{89}{27}$ and the fifth one gives $z = \pm\frac{5\sqrt{29}}{9}$.

Hence, we have found the following potential absolute extrema:

$$(-3, -7, 0),\ (3, 1, 0),\ \left(-\frac{2}{9},\ -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right),\ \left(-\frac{2}{9}, -\frac{89}{27}, \frac{5\sqrt{29}}{9}\right).$$

Calculating the value of f at these points we find that $f\left(-\frac{2}{9},\ -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right) = -\frac{85}{27}$ is the minimum and $f(3, 1, 0) = 28$ is the maximum.

9. We apply the method of Lagrange multipliers. We set $g(x, y) = x^2 + y^2 - 4$ and then we have to look for the extrema of f on the compact set $E = \{(x, y) \in \mathbb{R}^2 \colon g(x, y) \leq 0\}$.

We have that

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \iff \begin{cases} 8x = 0 \\ 20y = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

hence the only critical point is $(0, 0)$ and then we only have to determine the points (x, y) such that $g(x, y) = 0$ and $\nabla g(x, y) = 0$. This gives the following system of equations:

$$\begin{cases} 8x = 2\lambda x \\ 20y = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

From the first equation we get $x = 0$ or $\lambda = 4$.

- If $x = 0$, then the third equation gives $y = \pm 2$.
- If $\lambda = 4$, then the second equation gives $y = 0$.

Thus we have to check the value of f at the following four points: $(0,2),(0,-2),(2,0),(-2,0)$. We have that

$$\begin{cases} f(0,0)=0 \\ f(2,0)=f(-2,0)=16 \\ f(0,2)=40 \end{cases} \implies \begin{cases} f(0,0)=0 \text{ is the minimum} \\ f(0,2)=40 \text{ is the maximum.} \end{cases}$$