EXERCISE SHEET 8

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
April 7, 2025

Exercise 1. Let $G : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $G(x, y) = \big(\cos(x)+\sin(y)\,,\, -\sin(x)+\cos(y)\,,\, 2\sin(x)\cos(y)\big)^\top$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(u, v, w) = u^2 + v^2 + w\,.$

(i) Compute $J_{f\circ G}$ via matrix products, that is, exploit the chain rule.

(ii) Compute $J_{f\circ G}$ directly by substituting G in f.

Exercise 2. Suppose that G is the map that corresponds to a change of coordinates to spherical coordinates,[1] that is, use G to express (x, y, z) in spherical coordinates.

i) Verify that $(x, y, z) = G(\rho, \theta, \varphi)$ lies on the sphere of radius ρ for all θ and φ.

ii) Write the point $(x, y, z) = (\sqrt{6}, \sqrt{2}, -2\sqrt{2})$ in spherical coordinates.

iii) Compute the Jacobian matrix J_G of G for $\rho > 0$, $\theta \in \,]0, \pi[$ et $\varphi \in \,]0, 2\pi[$.

iv) Compute the Jacobian $\det(J_G)$ of G.

v) Noting that $(J_G)^\top \cdot J_G$ is a diagonal matrix, determine the inverse matrix $(J_G)^{-1}$.

Exercise 3. Let $D \subset \mathbb{R}^2$ and define $H\colon D \to \mathbb{R}^2$ by $(u, v) = H(x, y) = \big(H_1(x, y), H_2(x, y)\big)$ with

$$H_1(x, y) = \frac{y}{x + 2}\,, \qquad H_2(x, y) = \frac{x}{2y + 1}\,.$$

Determine the largest possible open set D such that the following conditions are satisfied:

- $H : D \to \mathbb{R}^2$ is injective,
- H is of class C^1,
- the image $\tilde{D} = H(D)$ is an open subset of $\mathbb{R}^2$,
- the inverse function $G = H^{-1} : \tilde{D} \to D$ is also of class C^1.

[1]Hint: the function $G\colon \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi[\longrightarrow \mathbb{R}^3 \setminus \{0\}$ is defined by

$$G(\rho, \theta, \varphi) = \big(G_1(\rho, \theta, \varphi), G_2(\rho, \theta, \varphi), G_3(\rho, \theta, \varphi)\big),$$

where

$$x = G_1(\rho, \theta, \varphi) = \rho \sin(\theta)\cos(\varphi),$$
$$y = G_2(\rho, \theta, \varphi) = \rho \sin(\theta)\sin(\varphi),$$
$$z = G_3(\rho, \theta, \varphi) = \rho \cos(\theta).$$

Explicitly compute the inverse function G.

Hint 1: Try computing the inverse function $G = H^{-1}$ first. Then use the above conditions to determine the appropriate domain D.

Hint 2: You may use the following fact: A continuously differentiable function is locally invertible near a point if the determinant of its Jacobian at that point is non-zero..

Exercise 4. Let $g : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \to g(x, y, z)$ be a C^1 function, $h : \mathbb{R}^2 \to \mathbb{R}^3$ given by $h(u, v) = (ve^{-2u},\ u^2e^{-v},\ u)$ and define $f = g \circ h$. Prove that

$$\frac{\partial f}{\partial u}(1,0) = 2\frac{\partial g}{\partial y}(0,1,1) + \frac{\partial g}{\partial z}(0,1,1).$$

Exercise 5. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (g_1(x, y), g_2(x, y))$ be a C^1 function, $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by $h(u, v) = (u + v, u - v)$ and define $f = g \circ h$. If

$$\mathbf{J}_f(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

then calculate $\frac{\partial g_1}{\partial x}(2, 0)$, $\frac{\partial g_1}{\partial y}(2, 0)$, $\frac{\partial g_2}{\partial x}(2, 0)$ and $\frac{\partial g_2}{\partial y}(2, 0)$.

Exercise 6. Prove that the electric field $\mathbf{E}$ of a charge Q (see Example 5.8 in the lecture notes) is conservative.

Exercise 7. (a) Let $\mathbf{F}(x, y) = P(x, y)\mathbf{e_1} + Q(x, y)\mathbf{e_2}$ be a conservative vector field. Show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

(b) Use part (a) to show that the following vector fields are not conservative:

i) $\mathbf{F}(x, y) = (2x \sin(2y) - 3y^2)\mathbf{e_1} + (2 - 6xy + 3x^2 \cos(2y))\mathbf{e_2}$.

ii) $\mathbf{G}(x, y) = (2 - 6xy + y^3)\mathbf{e_1} + (x^2 - 8y + 3xy^2)\mathbf{e_2}$.

Exercise 8. Find the maximum and minimum values of $f(x, y, z) = 3x^2 + y$, under the constraints $4x - 3y = 9$, $x^2 + z^2 = 9$.

Exercise 9. Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ under the constraint $x^2 + y^2 \le 4$.