EXERCISE SHEET 7 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
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1. (i) For $m \in \mathbb{R}$, let $x \to f_m(x)$ be the restriction of f to the line $y = mx$, that is
$f_m(x) = (mx - x^2)(mx - 2x^2) = 2x^4 - 3mx^3 + m^2x^2$,
$f'_m(x) = 8x^3 - 9mx^2 + 2m^2x$, $f'_m(0) = 0$,
$f''_m(x) = 24x^2 - 18mx + 2m^2$, $f''_m(0) = 2m^2$.

So 0 is a local minimum point of f_m if $m \neq 0$. Moreover, $f_0(x) = 2x^4$, such that 0 is also a local minimum point of f_0. Finally the restriction of f to the line $x = 0$ is the function y^2, which also has a local minimum in 0.

(ii) No, $(0,0)$ is not a local minimum point of f. Indeed any neighborhood of $(0,0)$ contains points where f is smaller than $0 = f(0,0)$:
$f(t, \frac{3}{2}t^2) = (\frac{3}{2}t^2 - t^2)(\frac{3}{2}t^2 - 2t^2) = -\frac{1}{4}t^4 < 0$ for all $t \neq 0$.

2. We look for the extrema of $f(x,y,z) = z$ under the constraint

$$g(x,y,z) = 4x^2 + 3y^2 + 2yz + 3z^2 - 4x - 1 = 0.$$

let $\Gamma = \{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 0\}$, be compact (and non-empty). To verify in particular that Γ is bounded, let us complete the square for g with respect to x:

$$g(x,y,z) = 4\left(x - \frac{1}{2}\right)^2 + 3y^2 + 2yz + 3z^2 - 2$$

and then the square with respect to y:

$$g(x,y,z) = 4\left(x - \frac{1}{2}\right)^2 + 3\left(y + \frac{1}{3}z\right)^2 + \frac{8}{3}z^2 - 2.$$

Thus, if $g(x,y,z) = 0$, then

$$4\left(x - \frac{1}{2}\right)^2 + 3\left(y + \frac{1}{3}z\right)^2 + \frac{8}{3}z^2 = 2$$

and Γ is therefore bounded. Since the function f is continuous on $\mathbb{R}^3$, the constraint on Γ is also continuous and, Γ is compact, it attains its maximum and minimum.

Note that $\nabla g(x,y,z) = (8x-4, 6y+2z, 2y+6z)^\top = (0,0,0)^\top \Leftrightarrow (x,y,z) = \left(\frac{1}{2},0,0\right)$, but $g\left(\frac{1}{2},0,0\right) = -2 \neq 0$ and therefore $\nabla g(x,y,z)$ does not vanish on Γ.

Let's apply the Lagrange multipliers method. We now have to find $x,y,z,\lambda \in \mathbb{R}$ such that

$$\begin{cases} f'_x = 0 = \lambda(8x-4) & (1)\\ f'_y = 0 = \lambda(6y+2z) & (2)\\ f'_z = 1 = \lambda(2y+6z) & (3)\\ 4x^2+3y^2+2yz+3z^2-4x-1=0 & (4)\end{cases}$$

Observe that $\lambda \neq 0$ because of (3). By (1) we have then $x = \frac{1}{2}$ and by (2) we have $z = -3y$. Substituting this into (4) gives

$$0 = 1 + 3y^2 - 6y^2 + 27y^2 - 2 - 1 = 24y^2 - 2 \Rightarrow y = \pm\frac{\sqrt{3}}{6}.$$

Thus the candidates for global extremum points are

$$(x,y,z) \in \left\{ \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2}\right) \right\}$$

and the maximal and minimal values of z are $\frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{2}$; they are realized at the points $\left(\frac{1}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2}\right)$.

3. (i) We look for the extrema of the function $f(x,y) = x^3+y^3$ under the constraint $g(x,y) = x^4+y^4-32 = 0$. Let $\Gamma = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$ be a compact set. Since $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on $\mathbb{R}^2$, the constraint $f|_\Gamma$ is continuous and therefore $f|_\Gamma$ attains its global maximum and minimum on the compact Γ (non empty). Note that $\nabla g(x,y,z) = (4x^3, 4y^3)^\top = (0,0)^\top \Leftrightarrow (x,y) = (0,0)$, but that $g(0,0) \neq 0$ and therefore $\nabla g(x,y) \neq (0,0)^\top$ for all (x,y) satisfying $g(x,y) = 0$. we can therefore introduce a Lagrange multiplier $\lambda \in \mathbb{R}$. Taking into account the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$, we obtain the system for the unknowns x,y,λ

$$\begin{cases} 3x^2 = 4\lambda x^3 \Leftrightarrow x^2(3-4\lambda x) = 0 & (1)\\ 3y^2 = 4\lambda y^3 \Leftrightarrow y^2(3-4\lambda y) = 0 & (2)\\ x^4+y^4-32 = 0 & (3)\end{cases}$$

From (1) and (2) we find several solutions:

$$(1) \quad \Leftrightarrow \quad x = 0 \text{ or } \lambda x = \frac{3}{4} \qquad \text{and} \qquad (2) \quad \Leftrightarrow \quad y = 0 \text{ or } \lambda y = \frac{3}{4}$$

- If $x = y = 0$, (3) is not satisfied, therefore impossible.
- If $x = 0$, then (3) implies that $y = \pm\sqrt[4]{32} = \pm 2\sqrt[4]{2}$. Then there exists a value of λ which satisfies (2).

- If $y = 0$, then $x = \pm 2\sqrt[4]{2}$ and (1) can be satisfied.
- If none of the variables are zero, then $x = y = \frac{3}{4\lambda}$ by (1) and (2). By (3) it follows that

$$x^4 + x^4 - 32 = 0, \ x = \pm 2, \ (x, y) = \pm(2, 2).$$

Candidates for global extremum points are

$$(x, y) \in \left\{(0, 2\sqrt[4]{2}), (0, -2\sqrt[4]{2}), (2\sqrt[4]{2}, 0), (-2\sqrt[4]{2}, 0), (2, 2), (-2, -2)\right\}$$

and we have the following table

(x, y)	$(0, 2\sqrt[4]{2})$	$(0, -2\sqrt[4]{2})$	$(2\sqrt[4]{2}, 0)$	$(-2\sqrt[4]{2}, 0)$	$(2, 2)$	$(-2, -2)$
$f(x, y)$	$8 \cdot 2^{3/4}$	$-8 \cdot 2^{3/4}$	$8 \cdot 2^{3/4}$	$-8 \cdot 2^{3/4}$	16	-16

Since $2^{3/4} < 2$, the maximum value of f is 16, attained at $(2, 2)$, and the minimum value is -16, attained at $(-2, -2)$.

(ii) Let $g(x, y, z) = x^2 + y^2 + z^2 - 4$ for $(x, y, z) \in \mathbb{R}^3$ and the closed ball $E = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) \leq 0\}$. The boundary of E is given by the set $\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$. Note that we have $\nabla g(x, y, z) = (2x, 2y, 2z)^\top = (0, 0, 0)^\top \ \Leftrightarrow \ x = y = z = 0$, but $g(0, 0, 0) = -4 \neq 0$ and therefore $\nabla g \neq (0, 0, 0)^\top$ on the boundary of E. We start by finding the extrema of f in $\mathring{E}$. The stationary points of f satisfy

$$\begin{cases} f'_x = 2x - 2 = 0 \\ f'_y = 2y + 2 = 0 \\ f'_z = 2z - 1 = 0 \end{cases} \Rightarrow \quad (x, y, z) = \left(1, -1, \frac{1}{2}\right) \quad \text{is the only stationary point}$$

which is indeed an interior point of E because $1^2 + (-1)^2 + \left(\frac{1}{2}\right)^2 = \frac{9}{4} < 4$.

To find the extrema of f on the boundary of E, let's apply the Lagrange multipliers method. We have to find $x, y, z, \lambda \in \mathbb{R}$ such that

$$\begin{cases} 2x - 2 = 2\lambda x & (1) \\ 2y + 2 = 2\lambda y & (2) \\ 2z - 1 = 2\lambda z & (3) \\ x^2 + y^2 + z^2 - 4 = 0 & (4) \end{cases}$$

Since $\lambda \neq 1$ (otherwise (1) to (3) are not satisfied), we can divide by $1 - \lambda$ to get from (1) to (3)

$$x = \frac{1}{1 - \lambda}, \qquad y = -\frac{1}{1 - \lambda}, \qquad z = \frac{1}{2(1 - \lambda)}$$

which we then substitute into (4) to get

$$\frac{2}{(1 - \lambda)^2} + \frac{1}{4(1 - \lambda)^2} - 4 = 0 \quad \Leftrightarrow \quad (1 - \lambda)^2 = \frac{9}{16} \quad \Leftrightarrow \quad 1 - \lambda = \pm\frac{3}{4}.$$

Thus we have $x_1 = -\dfrac{4}{3}, \quad y_1 = \dfrac{4}{3}, \quad z_1 = -\dfrac{2}{3}$ and $x_2 = \dfrac{4}{3}, \quad y_2 = -\dfrac{4}{3}, \quad z_2 = \dfrac{2}{3}$.
We calculate the values of f that are the potential extrema on E

(x,y,z)	$\left(1,-1,\frac{1}{2}\right)$	$\left(-\frac{4}{3},\frac{4}{3},-\frac{2}{3}\right)$	$\left(\frac{4}{3},-\frac{4}{3},\frac{2}{3}\right)$
$f(x,y,z)$	$-\frac{7}{2}$	$\frac{35}{4}$	$-\frac{13}{4}$

Thus the minimum of f on E is $-\frac{7}{2}$, attained at $\left(1,-1,\frac{1}{2}\right)$, and the maximum is $\frac{35}{4}$, attained at $\left(-\frac{4}{3},\frac{4}{3},-\frac{2}{3}\right)$.

4. We have

$$\int 3t\,dt = \frac{3}{2}t^2 + c_1, \quad \int -4e^{-t}\,dt = 4e^{-t} + c_2 \quad \text{and} \quad \int 12t^2\,dt = 4t^3 + c_3,$$

for some $c_1, c_2, c_3 \in \mathbb{R}$, hence $\mathbf{v}(t) = (\frac{3}{2}t^2 + c_1, 4e^{-t} + c_2, 4t^3 + c_3)^{\mathrm{T}}$ and using that $\mathbf{v}(0) = (0,1,-3)^{\mathrm{T}}$, we have $c_1 = 0$, $c_2 = -3$ and $c_3 = -3$. Therefore,

$$\mathbf{v}(t) = \left(\frac{3}{2}t^2, 4e^{-t} - 3, 4t^3 - 3\right)^{\mathrm{T}}.$$

Similarly, we find

$$\mathbf{r}(t) = \left(\frac{1}{2}t^3 - 5, -4e^{-t} - 3t + 6, t^4 - 3t - 3\right)^{\mathrm{T}}.$$

5 Consider two open intervals I and J, and functions

$$f : \begin{cases} J \times I \to \mathbb{R}, \\ (x,t) \to f(x,t), \end{cases}$$

and $a, b : I \to \mathbb{R}$, all of class C^1. Suppose that $\mathrm{Im}(a) \subset J$ and $\mathrm{Im}(b) \subset J$, and let

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx \tag{1}$$

for $t \in I$. Then F is of class C^1 and its derivative is

$$F'(t) = f\big(b(t),t\big) \cdot b'(t) - f\big(a(t),t\big) \cdot a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,dx\ . \tag{2}$$

Now we will apply the above to the questions at hand

i) We have

$$f(x,t) = \frac{x^t + \sin(x)}{\ln(x)} = \frac{e^{t\ln(x)} + \sin(x)}{\ln(x)} \in C^1\big(]1,\infty[\times]1,\infty[\big),$$

$a(t) = 2$ and $b(t) = 3$ with $a, b \in C^1(]1, \infty[)$. Since the bounds are constant, the right-hand side of (2) consists only of the integral and we have for all $t \in]1, \infty[$

$$F'(t) = \int_2^3 \frac{\partial}{\partial_t} \left(\frac{e^{t \ln(x)} + \sin(x)}{\ln(x)} \right) dx = \int_2^3 \frac{x^t \ln(x)}{\ln(x)} \, dx = \int_2^3 x^t \, dx = \left[\frac{x^{t+1}}{t+1} \right]_{x=2}^{x=3} = \frac{3^{t+1} - 2^{t+1}}{t+1} \, .$$

ii) We have $f(x, t) = \ln(x^2 + t^2) \in C^1\big(\mathbb{R} \times]1, \infty[\big)$, $a(t) = t$ and $b(t) = t^2$ with $a, b \in C^1(]1, \infty[)$. The bounds depend on t. We have

$$\begin{aligned} F'(t) &= \ln\Big(\big(t^2\big)^2 + t^2 \Big) \cdot \frac{d\left(t^2\right)}{dt} - \ln\big(t^2 + t^2\big) \cdot \frac{d(t)}{dt} + \int_t^{t^2} \frac{\partial}{\partial_t} \left(\ln(x^2 + t^2) \right) dx \\ &= 2t \ln\big(t^2 \left(t^2 + 1\right)\big) - \ln\left(2t^2\right) + \int_t^{t^2} \frac{2t}{x^2 + t^2} \, dx \, . \end{aligned}$$

Since $2t \displaystyle\int_t^{t^2} \frac{1}{x^2 + t^2} \, dx = \left[\frac{2t}{t} \arctan\left(\frac{x}{t}\right) \right]_{x=t}^{x=t^2} = 2\arctan(t) - \frac{\pi}{2}$, we finally have

$$F'(t) = 2t \ln\left(t^2(t^2 + 1)\right) - \ln\left(2t^2\right) + 2\arctan(t) - \frac{\pi}{2} \, .$$

6. These functions are again of the form (1).

i) Here we have $f(x, t) = \frac{\sin(\cos(tx))}{x} \in C^1\big( ]0, \infty[\times]0, \infty[\big)$, $a(t) = \sqrt{t}$ and $b(t) = \frac{1}{t}$ with $a, b \in C^1( ]0, \infty[)$. So

$$\begin{aligned} F'(t) &= \frac{\sin\left(\cos\left(t\frac{1}{t}\right)\right)}{\frac{1}{t}} \cdot \frac{d}{dt}\left(\frac{1}{t}\right) - \frac{\sin(\cos(t\sqrt{t}))}{\sqrt{t}} \cdot \frac{d}{dt}(\sqrt{t}\,) + \int_{\sqrt{t}}^{1/t} \frac{\partial}{\partial t} \left(\frac{\sin(\cos(tx))}{x} \right) dx \\ &= -\frac{\sin(\cos(1))}{t} - \frac{\sin(\cos(t^{3/2}))}{2t} + \int_{\sqrt{t}}^{1/t} \cos\left(\cos(tx)\right)\left(-\sin(tx)\right) dx \\ &= -\frac{\sin(\cos(1))}{t} - \frac{\sin(\cos(t^{3/2}))}{2t} + \left[\frac{\sin(\cos(tx))}{t} \right]_{x=\sqrt{t}}^{x=1/t} = -\frac{3\sin(\cos(t^{3/2}))}{2t} \, . \end{aligned}$$

ii) For $f(x, t) = \frac{e^{tx^3}}{x} \in C^1( ]0, \infty[\times]0, \infty[\, ,)$, $a(t) = 1$ and $b(t) = \sqrt[3]{t}$ with $a, b \in C^1( ]0, \infty[ )$, only the upper bound depends on t so that the term in $f(a(t), t)$ does not appear.

$$\begin{aligned} F'(t) &= \left. \frac{e^{tx^3}}{x} \right|_{x=\sqrt[3]{t}} \cdot \frac{d}{dt}\left(\sqrt[3]{t}\right) + \int_1^{\sqrt[3]{t}} \frac{\partial}{\partial t} \left(\frac{e^{tx^3}}{x} \right) dx \\ &= \frac{e^{t^2}}{\sqrt[3]{t}} \cdot \frac{1}{3} t^{-2/3} + \int_1^{\sqrt[3]{t}} x^2 e^{tx^3} \, dx = \frac{1}{3} \frac{e^{t^2}}{t} + \left[\frac{1}{3t} e^{tx^3} \right]_{x=1}^{x=\sqrt[3]{t}} \\ &= \frac{1}{3t}\left(e^{t^2} - e^t\right) + \frac{1}{3}\frac{e^{t^2}}{t} = \frac{1}{3t}\left(2e^{t^2} - e^t\right). \end{aligned}$$

7.

i) Let $f(x,y) = \dfrac{x^3y^2}{x^4+y^8}$ for $(x,y) \neq (0,0)$. Then

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^3\cdot 0^2}{x^4+0^8} = \lim_{x\to 0} 0 = 0,$$

while

$$\lim_{y\to 0} f(y^2,y) = \lim_{y\to 0} \frac{y^6y^2}{y^8+y^8} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}.$$

Therefore the limit does not exist.

ii) Consider

$$f(x,y) = \frac{x^4y}{x^4+y^8} \quad \text{for} \quad (x,y)\neq(0,0) \quad \text{and} \quad g(u,v) = \frac{u^2\sqrt[4]{|v|}}{u^2+v^2} \quad \text{for} \quad (u,v)\neq(0,0).$$

Then $f(x,y) = \pm g(x^2,y^4)$ for all $(x,y)\neq(0,0)$.

Now check that $\lim_{(u,v)\to(0,0)} g(u,v) = 0$. We have, for all $(u,v)\neq(0,0)$,

$$0 \leq g(u,v) \leq \frac{||(u,v)||^2\sqrt[4]{||(u,v)||}}{u^2+v^2} = \sqrt[4]{||(u,v)||}$$

with $\lim_{(u,v)\to(0,0)} \sqrt[4]{||(u,v)||} = 0$. The squeeze theorem ensures that $\lim_{(u,v)\to(0,0)} g(u,v) = 0$. As such

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{(x,y)\to(0,0)} g(x^2,y^4) = \lim_{(u,v)\to(0,0)} g(u,v) = 0$$

and so $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.