EXERCISE SHEET 6 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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1. i) We obtain the stationary points of the function f by solving the system

$$\nabla f(x,y) = 0 \quad \Leftrightarrow \quad \begin{cases} f'_x(x,y) = 2x - 2 = 0 \\ f'_y(x,y) = 2y - 1 = 0 \end{cases} \quad \Rightarrow \quad (x,y) = \left(1, \tfrac{1}{2}\right).$$

The only stationary point of f is therefore $\left(1, \frac{1}{2}\right)$. Since $f''_{xx}(x,y) = 2$ and

$$\det(\mathrm{Hess}_f(x,y)) = \det\begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4,$$

we have that $f''_{xx}(1, \frac{1}{2}) = 2$ and $\det(\mathrm{Hess}_f(1, \frac{1}{2})) > 0$. Therefore f reaches a local minimum at the point $(1, \frac{1}{2})$ where it is $f\left(1, \frac{1}{2}\right) = -\frac{1}{4}$.

Note: In fact f admits a global minimum in $(1, 1/2)$ because, for all $(x,y) \in \mathbb{R}^2$, $f(x,y) = (x-1)^2 + (y - \frac{1}{2})^2 - \frac{1}{4} \geq -\frac{1}{4} = f(1, 1/2)$.

ii) The system

$$\begin{cases} f_x(x,y) = -\sin(x) = 0 \\ f_y(x,y) = 6y = 0 \end{cases}$$

gives the stationary points $(x,y) = (k\pi, 0)$ with $k \in \mathbb{Z}$. Since

$$\det(\mathrm{Hess}_f(x,y)) = \det\begin{pmatrix} -\cos(x) & 0 \\ 0 & 6 \end{pmatrix} = -6\cos(x),$$

we have

$$\det(\mathrm{Hess}_f(k\pi, 0)) = \begin{cases} -6, & k \text{ even} \\ 6, & k \text{ odd} \end{cases}$$

The points $(k\pi, 0)$ with k even are therefore saddle points with $f(k\pi, 0) = 3$. For k odd, the equality $f''_{xx}(k\pi, 0) = -\cos(k\pi) = 1 > 0$ implies that f admits local minima at points $(k\pi, 0)$ with $f(k\pi, 0) = 1$.

2. i) We solve the system

$$\begin{cases} f'_x(x,y,z) = & -4x & + & 4y & & & = 0 \\ f'_y(x,y,z) = & 4x & - & 10y & + & 2z & = 0 \\ f'_z(x,y,z) = & & & 2y & - & 2z & = 0 \end{cases}$$

to obtain the only stationary point $(0,0,0)$ (for example, by "adding" the three equations, we immediately have $-4y = 0$). Then we compute the determinant of the Hessian $D_3(x,y,z)$ and the two sub-determinants $D_2(x,y,z)$ and $D_1(x,y,z)$ (this is in fact the first coefficient of the first line):

$$D_3(x,y,z) = \det\begin{pmatrix} -4 & 4 & 0 \\ 4 & -10 & 2 \\ 0 & 2 & -2 \end{pmatrix} = -32, \qquad D_2(x,y,z) = \det\begin{pmatrix} -4 & 4 \\ 4 & -10 \end{pmatrix} = 24$$

$$\text{and} \quad D_1(x,y,z) = -4,$$

hence

$$D_1(0,0,0) = -4 < 0, \qquad D_2(0,0,0) = 24 > 0 \qquad \text{and} \qquad D_3(0,0,0) = -32 < 0.$$

Therefore, the function f has a local maximum at $(0,0,0)$ and $f(0,0,0) = 2$.

ii) Since the function f admits partial derivatives everywhere in $\mathring{D}$ (the interior of D), its absolute extrema are found among the stationary points in the interior of D or the points on the boundary of D.

Stationary points in the interior of D:

$$\begin{cases} f'_x(x,y) = 2x - y - 1 = 0 \\ f'_y(x,y) = -x + 2y - 1 = 0 \end{cases} \qquad \Rightarrow \qquad (x,y) = (1,1) \in \mathring{D}.$$

Since

$$D_2(x,y) = \det(\mathrm{Hess}_f(x,y)) = \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3 > 0 \qquad \text{and} \qquad D_1(x,y) = f''_{xx}(x,y) = 2 > 0\,,$$

the point $(1,1)$ is a local minimum of f. Moreover, we have $f(1,1) = -1$.

On the boundary of D we have:

Note first that the boundary of D is the union of the following three subsets of $\mathbb{R}^2$:

$$\{(x,0) : 0 \le x \le 3\} \cup \{(0,y) : 0 \le y \le 3\} \cup \{(x, 3-x) : 0 \le x \le 3\}.$$

The evaluation of the function $f(x,y) = x^2 - xy + y^2 - x - y$ on the boundary gives

$$f(x,0) = x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\,, \qquad 0 \le x \le 3\,,$$

$$f(0,y) = y^2 - y = \left(y - \frac{1}{2}\right)^2 - \frac{1}{4}\,, \qquad 0 \le y \le 3\,,$$

$$f(x,3-x) = 3(x^2 - 3x + 2) = 3\left[\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}\right], \qquad 0 \le x \le 3\,.$$

The idea now is to look for the extrema of these one-dimensional functions in the specified interval which are either at the stationary points or at the extremities of the interval (cf. Analysis I). Let us first use the notation $g(x) = f(x,0)$. Then $g'(x) = 2\left(x - \frac{1}{2}\right) = 0 \Leftrightarrow x = \frac{1}{2}$ and $g\left(\frac{1}{2}\right) = -\frac{1}{4}$. Since $g''(x) = 2 > 0$, g has a local minimum at $x = \frac{1}{2}$. Furthermore, we have $g(0) = 0$ and $g(3) = 6$. We therefore have

$$\max_{0 \le x \le 3} f(x,0) = f(3,0) = 6 \qquad \text{and} \qquad \min_{0 \le x \le 3} f(x,0) = f\left(\frac{1}{2}, 0\right) = -\frac{1}{4}.$$

Similarly, we look for the extrema of the functions $h(y) = f(0, y)$ and $k(x) = f(x, 3 - x)$. The function h has exactly the same behavior as g and for k we have

$$k'(x) = 6\left(x - \tfrac{3}{2}\right) = 0 \quad \Leftrightarrow \quad x = \tfrac{3}{2}, \qquad k\left(\tfrac{3}{2}\right) = -\tfrac{3}{4},$$

$$k''(x) = 6 > 0 \quad (\Rightarrow \text{ local minimum}), \qquad k(0) = k(3) = 6,$$

so that we obtain

$$\max_{0\le y\le 3} f(0, y) = f(0, 3) = 6\,, \qquad \min_{0\le y\le 3} f(0, y) = f\left(0, \tfrac{1}{2}\right) = -\tfrac{1}{4}\,,$$

$$\max_{0\le x\le 3} f(x, 3-x) = f(3, 0) = f(0, 3) = 6\,, \qquad \min_{0\le x\le 3} f(x, 3-x) = f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) = -\tfrac{3}{4}\,.$$

It follows that f admits an absolute minimum at $(1, 1)$ of value $f(1, 1) = -1$ and absolute maximums at $(3, 0)$ and at $(0, 3)$ of values $f(3, 0) = f(0, 3) = 6$.

3. Since the partial derivatives of f exist on $\mathring{D}$, the absolute extrema are reached at stationary points in the interior of D or at points on the boundary of D. Since $f'_y = -1$ never vanishes on D, the function f has no stationary points.

Since D is a rectangular parallelepiped parallel to the axes, we can determine the behavior of f on the boundary of D by examining its partial derivatives. For $(x, y, z) \in D$ we have:

$f'_x = z + 1 > 0 \quad \Rightarrow \quad f$ is strictly increasing in the direction of x and therefore maximal at $x = a$ and minimal at $x = 0$.

$f'_y = -1 < 0 \quad \Rightarrow \quad f$ is strictly decreasing in the direction of y and therefore maximal at $y = 0$ and minimal at $y = b$.

$f'_z = x + 2 > 0 \quad \Rightarrow \quad f$ is strictly increasing in the direction of z and therefore maximal at $z = c$ and minimal at $z = 0$.

The function f has its absolute maximum at $(a, 0, c)$ and its absolute minimum at $(0, b, 0)$.

In order to calculate the extremal values of f, we need to find its expression. From the given partial derivatives, we obtain successively

$$f'_x(x, y, z) = -1 \quad \Rightarrow \quad f(x, y, z) = -y + g(x, z) \quad \Rightarrow \quad f'_x(x, y, z) = g'_x(x, z) = z + 1$$

$$\Rightarrow \quad g(x, z) = (z + 1)x + h(z) \qquad \Rightarrow \qquad f'_z(x, y, z) = x + h'(z) = x + 2$$

$$\Rightarrow \quad h(z) = 2z + C, \quad C \in \mathbb{R} \qquad \Rightarrow \qquad g(x, z) = (z + 1)x + 2z + C$$

$$\Rightarrow \quad f(x, y, z) = -y + (z + 1)x + 2z + C$$

The condition $f(0, 0, 0) = 3$ then implies that $C = 3$ and $f(x, y, z) = (z+1)x - y + 2z + 3$. Thus the absolute maximum of f is $f(a, 0, c) = a(c+1)+2c+3$ and its absolute minimum is $f(0, b, 0) = 3-b$.

Note: We could also have calculated the expression of f from the start but the approach taken here is more instructive.

4. (i) For a function f of class C^1, the directional derivative $\nabla_{\mathbf{v}} f(\mathbf{a})$ at point $\mathbf{a}$ along the non-zero vector $\mathbf{v}$ is given by

$$\nabla_{\mathbf{v}} f(\mathbf{a}) = \langle \nabla f(\mathbf{a}), \mathbf{v} \rangle\,.$$

The given function f is indeed C^1. Since

$$\nabla f(x,y,z) = (yz, xz, xy)^T \qquad \text{and} \qquad \nabla f(1,-1,2) = (-2,2,-1)^T\,,$$

we obtain

$$\nabla_{\mathbf{v}} f(1,-1,2) = (-2,2,-1)^T \cdot (2,-1,2)^T = -8 \;\text{ and }\; \nabla_{\mathbf{e}} f(1,-1,2) = -\frac{8}{3}\,.$$

(ii) The slope of f in $\mathbf{a}$ in the direction of the unit vector $\boldsymbol{u}$ is given by the directional derivative along this unit vector, that is, by

$$\begin{aligned}\nabla_{\mathbf{u}} f(\mathbf{a}) = \langle \nabla f(\mathbf{a}), \mathbf{u}\rangle &= (-2,2,-1)^T \cdot \big(\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)\big)^T \\ &= 2\sin(\theta)\big(\sin(\varphi) - \cos(\varphi)\big) - \cos(\theta) =: g(\theta,\varphi)\,,\end{aligned}$$

where $g : [0,\pi] \times [0, 2\pi[\subset \mathbb{R}^2 \to \mathbb{R}$.

(iii) We know that at a point where f is differentiable, the slope of the tangent to the graph is maximal in the direction of the gradient and is equal to the norm of the gradient.

Of course the slope of the tangent to the graph is minimal in the direction opposite to the gradient and is then equal to the opposite of the norm of the gradient.

At the point $\mathbf{a} = (1,-1,2)$, the maximum (minimum) slope is therefore

$$\|\nabla f(1,-1,2)\| = 3 \qquad \big(-\|\nabla f(1,-1,2)\| = -3\big).$$

The corresponding directions are $\pm\frac{\nabla f(1,-1,2)}{\|\nabla f(1,-1,2)\|} = \pm\frac{1}{3}(-2,2,-1)$. To find the angles $(\theta,\varphi) \in [0,\pi] \times [0,2\pi[$ we have to solve

$$\begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix} = \begin{pmatrix} \mp\frac{2}{3} \\ \pm\frac{2}{3} \\ \mp\frac{1}{3} \end{pmatrix},$$

that is to say, for the maximal slope,

$$\begin{aligned}\theta = \arccos\left(-\tfrac{1}{3}\right) &\overset{\theta\in[0,\pi]}{\Rightarrow} \sin(\theta) = \sqrt{1 - \left(-\tfrac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3} \quad\Rightarrow\quad \cos(\varphi) = -\tfrac{1}{\sqrt{2}} \text{ et } \sin(\varphi) = \tfrac{1}{\sqrt{2}} \\ &\overset{\varphi\in[0,2\pi[}{\Rightarrow} \varphi = \frac{3\pi}{4} \quad\Rightarrow\quad (\theta,\varphi) = \left(\arccos\left(-\frac{1}{3}\right), \frac{3\pi}{4}\right)\end{aligned}$$

and for the minimal slope,

$$\begin{aligned}\theta = \arccos\left(\tfrac{1}{3}\right) &\Rightarrow \sin(\theta) = \sqrt{1 - \left(\tfrac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3} \quad\Rightarrow\quad \cos(\varphi) = \tfrac{1}{\sqrt{2}} \text{ et } \sin(\varphi) = -\tfrac{1}{\sqrt{2}} \\ &\Rightarrow \varphi = \frac{7\pi}{4} \quad\Rightarrow\quad (\theta,\varphi) = \left(\arccos\left(\frac{1}{3}\right), \frac{7\pi}{4}\right)\end{aligned}$$

The slope of f at $\mathbf{a}$ is therefore maximal for the angles $(\theta,\varphi) = \left(\arccos\left(-\frac{1}{3}\right), \frac{3\pi}{4}\right)$ and minimal for $(\theta,\varphi) = \left(\arccos\left(\frac{1}{3}\right), \frac{7\pi}{4}\right)$.

5. Since the gradient of $f(x,y) = 4xy$ is $\nabla f(x,y) = \begin{pmatrix} 4y \\ 4x \end{pmatrix}$, the unique stationary point of f is $(0,0)$ and we have $H := \mathrm{Hess}_f(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$. What's more

$$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix},$$

which shows that 4 and -4 are eigenvalues and we can choose the corresponding (normalized) eigenvectors

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

For U, we can choose the following matrix whose columns are the two normalized eigenvectors:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

It is useful to remember that, U being orthogonal, implies that $U^{-1} = U^\top$. Hence

$$(\star) \quad \begin{pmatrix} x \\ y \end{pmatrix} = U\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \quad \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = U^\top \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x+y \\ -x+y \end{pmatrix}.$$

Let's check that U satisfies the statement of the problem. As $U^{-1}HU = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} := D$, it follows that

$$\begin{aligned} f(x,y) &= \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix} H \begin{pmatrix} x \\ y \end{pmatrix} \overset{(\star)}{=} \frac{1}{2}\begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix} U^\top H U \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix} D \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \frac{1}{2}\left(4\bar{x}^2 - 4\bar{y}^2\right) = \bar{f}(\bar{x},\bar{y}) \end{aligned}$$

and we have indeed that

$$\bar{f}(\bar{x},\bar{y}) = \frac{1}{2}\left(4\bar{x}^2 - 4\bar{y}^2\right).$$

What's more

$$f(x,y) = \frac{1}{2}\left(4\left(\frac{x+y}{\sqrt{2}}\right)^2 - 4\left(\frac{-x+y}{\sqrt{2}}\right)^2\right) = (x+y)^2 - (-x+y)^2 .$$

(Note that expanding $(x+y)^2 - (-x+y)^2$ yields $4xy$.) From this expression for f, it is easy to see that $(0,0)$ is a saddle point of f. We can also see this by noticing that $\det(\mathrm{Hess}_f(0,0)) = -16 < 0$.

6. (a) (i) Let $F(x,y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$. We have $\nabla F(p,q) = \left(\frac{2p}{a^2}, \frac{2q}{b^2}\right)$ and then the tangent line is given by the equation

$$\nabla F(p,q) \cdot \begin{pmatrix} x-p \\ y-q \end{pmatrix} = 0 \iff \left(\frac{2p}{a^2}, \frac{2q}{b^2}\right) \cdot \begin{pmatrix} x-p \\ y-q \end{pmatrix} = 0 \iff \frac{px}{a^2} - \frac{p^2}{a^2} + \frac{qy}{b^2} - \frac{q^2}{b^2} = 0.$$

Using that $\left(\frac{p}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, we obtain that

$$(\varepsilon_1): \qquad \frac{px}{a^2} + \frac{qy}{b^2} = 1.$$

(ii) Exactly as before, we can find that the equation of the tangent line is

$$(\varepsilon_2): \qquad \frac{px}{a^2} - \frac{qy}{b^2} = 1.$$

(b) (i) The line (ε_1) is parallel to the horizontal axes if and only if $p = 0$. Moreover, the point $(p, q) = (0, q)$ belongs to the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, thus, $q = \pm b$. Therefore, the set of points whose tangent is parallel to the horizontal axes is $\{(0, -b), (0, b)\}$ and the tangent line takes the form $y = -b$ (for the point $(0, -b)$) and $y = b$ (for the point $(0, b)$).

(ii) The line (ε_1) is parallel to the horizontal axes if and only if $p = 0$. The point $(0, q)$ belongs in the hyperbola $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$, hence $q^2 = -b^2$, which implies that $q = b = 0$, yielding a contradiction. Therefore, there is not tangent lines to the hyperbola that are parallel to the horizontal axes.

7. Let $F(x, y, z) = xz^2 - 2x^2y + y^2z$. Evaluating F at point $(1, 1, z_0)$ we have

$$F(1, 1, z_0) = 0 \quad \Leftrightarrow \quad z_0^2 - 2 + z_0 = 0 \quad \Leftrightarrow \quad z_0 = 1 \text{ or } z_0 = -2\,.$$

The equation of the tangent plane to the surfance $F(x, y, z) = 0$ at the point (x_0, y_0, z_0) is

$$(x - x_0, y - y_0, z - z_0)^\top \cdot \nabla F(x_0, y_0, z_0) = 0.$$

Since

$$\nabla F(x, y, z) = (\partial_x F(x, y, z),\, \partial_y F(x, y, z),\, \partial_z F(x, y, z))^\top = \left(z^2 - 4xy,\, -2x^2 + 2yz,\, 2xz + y^2\right)^\top,$$

we have for the point $(x_0, y_0, z_0) = (1, 1, 1)$

$$\nabla F(1, 1, 1) = (-3, 0, 3)^\top$$

and the equation of the tangent plane is

$$\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} = 0 \quad \Leftrightarrow \quad -3(x-1) + 0(y-1) + 3(z-1) = 0 \quad \Leftrightarrow \quad x - z = 0\,.$$

For $(x_0, y_0, z_0) = (1, 1, -2)$ we have

$$\nabla F(1, 1, -2) = (0, -6, -3)^\top$$

and the equation of the tangent plane is

$$\begin{pmatrix} x-1 \\ y-1 \\ z+2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -6 \\ -3 \end{pmatrix} = 0 \quad \Leftrightarrow \quad 0(x-1) - 6(y-1) - 3(z+2) = 0 \quad \Leftrightarrow \quad 2y + z = 0\,.$$

8. For $t \in \mathbb{R}$, consider $(x, y, z) = (27t^3, 1/t^2, 1/t)$, which satisfies the constraint $xyz = 27$. As

$$\lim_{t \to \pm\infty} f(27t^3.1/t^2.1/t) = \lim_{t \to \pm\infty} \left(27t^3 + t^{-2} + t^{-1}\right) = \pm\infty,$$

f does not reach a global extremum under the constraint $xyz - 27 = 0$.

Note. The set $\{(x, y, z) \in \mathbb{R}^3 : xyz - 27 = 0\}$ is closed but not bounded. It is therefore not compact.