EXERCISE SHEET 6

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
March 24, 2025

Exercise 1. Determine the stationary points of the following functions and find their nature:

i) $f(x,y) = x^2 + y^2 - 2x - y + 1$.
ii) $f(x,y) = 2 + 3y^2 + \cos(x)$.

Exercise 2. Determine the stationary points of the function $f(x,y) = x^2 - xy + y^2 - x - y$ on $D = \{(x,y) : x \geq 0,\ y \geq 0,\ x + y \leq 3\}$ and study its nature.

Exercise 3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class C^1 such that

$$\frac{\partial f}{\partial x} = z + 1, \qquad \frac{\partial f}{\partial y} = -1, \qquad \frac{\partial f}{\partial z} = x + 2.$$

Determine the absolute extrema of f under the constraint

$$D = \{(x,y,z) : 0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c\}, \quad \text{where } a, b, c > 0,$$

given $f(0,0,0) = 3$.

Exercise 4. Let $f(x,y,z) = xyz$ be a function and let $\mathbf{a} = (1,-1,2)$ be a point.

(i) Find the directional derivative of f at $\mathbf{a}$ along the vector $\mathbf{v} = (2,-1,2)^T$ and along the unit vector $\mathbf{e} = \frac{1}{3}(2,-1,2)^T$.
(ii) Let $\mathbf{u}$ be a unit vector expressed in spherical coordinates, that is,

$$\mathbf{u} = \big(\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)\big)^T.$$

Calculate the slope of f at $\mathbf{a}$ along the vector $\mathbf{u}$ as a function of (θ, φ).
(iii) Find the values of θ and φ for which the slope of f at $\mathbf{a}$ are maximal and minimal.

Exercise 5. Let $f(x,y) = 4xy$, $x, y \in \mathbb{R}$ be a function. Find the linear change of coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = U \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

such that U is an orthogonal matrix and the function $\bar{f}$ is defined by $\bar{f}(\bar{x},\bar{y}) = f(x,y)$ and has the form

$$\bar{f}(\bar{x},\bar{y}) = \frac{1}{2}\left(\lambda_1 \bar{x}^2 + \lambda_2 \bar{y}^2\right), \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

What is the nature of the stationary point $(0,0)$ of f?

Exercise 6. (a) Determine the tangent line to the following curves at the point (p, q):

(i) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

(ii) $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$

(b) Determine the sets of points in which the above lines are parallel to the horizontal axes.

Exercise 7. Determine the equation of the plane tangent to the surface $xz^2 - 2x^2y + y^2z = 0$ at points of the form $(1, 1, z_0)$.

Exercise 8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $f(x, y, z) = x + y + z$. Determine the points at which f reaches a global extremum under the constraint $xyz - 27 = 0$.

Exercise 9. (extra exercise)* Determine the stationary points of the function $f(x, y, z) = -2x^2 - 5y^2 - z^2 + 4xy + 2yz + 2$ on $D = \mathbb{R}^3$ and study its nature.

(Hint: use Sylvester's Criterion)