EXERCISE SHEET 5 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
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1. The tangent plane to the graph of f at the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$, is given by

$$\left\{(x,y,z) \in \mathbb{R}^3 : z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)\right\}.$$

To that end, we compute $\frac{\partial f}{\partial x}(x,y) = 3x^2y + 2x$ and $\frac{\partial f}{\partial y}(x,y) = x^3 + 2y$. Now observe that $f(1,1) = 3$, $\frac{\partial f}{\partial x}(1,1) = 5$ and $\frac{\partial f}{\partial y}(1,1) = 3$ such that the equation for the tangent plane at $(x_0, y_0) = (1,1)$ becomes

$$z = 3 + 5(x-1) + 3(y-1) \qquad \Leftrightarrow \qquad 5x + 3y - z = 5 \ .$$

2. Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$f(x,y,z) = -2\cos(\pi x) + x^2 y + 3e^{xz} + yz - 23.$$

Then, $f(3,2,0) = 0$. Besides, we have

$$\frac{\partial f}{\partial x}(x,y,z) = -2\pi\sin(\pi x) + 2xy + 3ze^{xz}, \quad \frac{\partial f}{\partial y}(x,y,z) = x^2 + z, \quad \frac{\partial f}{\partial z}(x,y,z) = 3xe^{xz} + y.$$

Therefore,

$$f(3,2,0) + \frac{\partial f}{\partial x}(3,2,0)(x-3) + \frac{\partial f}{\partial y}(3,2,0)(y-2) + \frac{\partial f}{\partial 2}(3,2,0)z = 12(x-3) + 9(y-2) + 11z.$$

This means that the equation for the tangent plane at $(3,2,0)$ is

$$12x + 9y + 11z = 54.$$

3. (a)

$$\frac{\partial f}{\partial x}(x,t) = -\frac{x}{2t} f(x,t)$$

and

$$\frac{\partial^2 f}{\partial x^2}(x,t) = -\frac{1}{2t} f(x,t) - \frac{x}{2t}\frac{\partial f}{\partial x}(x,t) = \left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right) f(x,t) = \frac{\partial f}{\partial t}(x,t).$$

(b) By the product rule and the result in (a) we get

$$\begin{aligned}
&\frac{\partial g}{\partial t}(x,y,t) - \frac{\partial^2 g}{\partial x^2}(x,y,t) - \frac{\partial^2 g}{\partial y^2}(x,y,t) = \\
&\frac{\partial f}{\partial t}(x,t)f(y,t) + f(x,t)\frac{\partial f}{\partial t}(y,t) - \frac{\partial^2 f}{\partial x^2}(x,t)f(y,t) - f(x,t)\frac{\partial^2 f}{\partial y^2}(y,t) = \\
&f(x,t)\left(\frac{\partial f}{\partial t}(y,t) - \frac{\partial^2 f}{\partial y^2}(y,t)\right) + f(y,t)\left(\frac{\partial f}{\partial t}(x,t) - \frac{\partial^2 f}{\partial x^2}(x,t)\right) = 0.
\end{aligned}$$

4. For $(x,y) \neq (0,0)$ the partial derivatives are

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\left(xy\,\frac{x^2-y^2}{x^2+y^2}\right) = xy\,\frac{4xy^2}{(x^2+y^2)^2} + y\,\frac{x^2-y^2}{x^2+y^2}\ , \\
\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}\left(xy\,\frac{x^2-y^2}{x^2+y^2}\right) = xy\,\frac{-4x^2y}{(x^2+y^2)^2} + x\,\frac{x^2-y^2}{x^2+y^2} = \frac{-4x^3y^2}{(x^2+y^2)^2} + x - \frac{2xy^2}{x^2+y^2}\ .
\end{aligned}$$

For $(x,y) = (0,0)$, we use the definition:

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0\,, \\
\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0\,.
\end{aligned}$$

To calculate the second (mixed) partial derivatives at $(0,0)$ we use the definition again:

$$\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y}(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0}\frac{-\frac{h^3}{h^2}-0}{h} = \lim_{h\to 0}\frac{(-h)-0}{h} = -1$$

$$\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1.$$

We observe that

$$\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y}(0,0) \neq \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(0,0).$$

For $(x,y) \neq (0,0)$,

$$\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(x,y) = -\frac{12x^2y^2}{(x^2+y^2)^2} + \frac{8x^3y^2}{(x^2+y^2)^3}\,2x + 1 - \frac{2y^2}{x^2+y^2} + \frac{2xy^2}{(x^2+y^2)^2}\,2x$$

exists and

$$\lim_{y\to 0}\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(0,y) = \lim_{y\to 0}\left(1 - \frac{2y^2}{y^2}\right) = \lim_{y\to 0}(-1) = -1 \neq 1 = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(0,0).$$

Thus, $\dfrac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}$ is not continuous at 0. In fact

$$\lim_{x\to 0}\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(x,0)=\lim_{x\to 0}1=1\neq -1,$$

So the limit $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(x,y)$ does not exist.

5.

$$\frac{\partial f}{\partial x}(x,y)=e^x\sin y \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y)=e^x\cos y,$$

thus we have

$$\frac{\partial^2 f}{\partial x^2}(x,y)=e^x\sin y,\qquad \frac{\partial^2 f}{\partial x\partial y}(x,y)=\frac{\partial^2 f}{\partial y\partial x}(x,y)=e^x\cos y \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}(x,y)=-e^x\sin y.$$

Then the Hessian matrix of f is

$$\mathrm{Hess}(f)(x,y)=\begin{pmatrix} e^x\sin y & e^x\cos y\\ e^x\cos y & -e^x\sin y\end{pmatrix}.$$

If we denote its determinant by $\det(x,y)$, then we have

$$\det(x,y)=-e^{2x}(\sin y)^2-e^{2x}(\cos y)^2=-e^{2x}((\sin y)^2+(\cos y)^2)=-e^{2x},$$

which does not depend on y, as it is a function only of x.

6. i) The Taylor polynomial $p_2(x,y)$ of order 2 around the origin of function $f(x,y)$ is given by

$$p_2(x,y)=f(0,0)+f'_x(0,0)\,x+f'_y(0,0)\,y+\frac{1}{2}f''_{xx}(0,0)\,x^2+f''_{xy}(0,0)\,xy+\frac{1}{2}f''_{yy}(0,0)\,y^2.$$

Here, we have

$$f(x,y)=x^2y+2xy+3y^2-5x+1\,,$$

$$f'_x(x,y)=2xy+2y-5\,, \qquad f'_y(x,y)=x^2+2x+6y\,,$$

$$f''_{xx}(x,y)=2y\,, \qquad f''_{xy}(x,y)=2x+2\,, \qquad f''_{yy}(x,y)=6\,,$$

where

$$f(0,0)=1,\quad f'_x(0,0)=-5,\quad f'_y(0,0)=0,\quad f''_{xx}(0,0)=0,\quad f''_{xy}(0,0)=2,\quad f''_{yy}(0,0)=6,$$

and therefore,

$$p_2(x,y)=1+(-5)\cdot x+0\cdot y+\frac{1}{2}\cdot 0\cdot x^2+2\cdot xy+\frac{1}{2}\cdot 6\cdot y^2=1-5x+2xy+3y^2.$$

the error $d^2 \cdot \varepsilon(x,y)$ must satisfy $\lim\limits_{(x,y)\to 0} \varepsilon(x,y) = 0$, where $d = \|(x,y)-(0,0)\| = \sqrt{x^2+y^2}$. Here we have

$$d^2 \cdot \varepsilon(x,y) = f(x,y) - p_2(x,y) = x^2y + 2xy + 3y^2 - 5x + 1 - (1 - 5x + 2xy + 3y^2) = x^2y$$

with,

$$0 \leq |\varepsilon(x,y)| = \frac{|x^2y|}{||(x,y)||^2} \leq \frac{||(x,y)||^3}{||(x,y)||^2} = ||(x,y)||,$$

for all $(x,y) \neq (0,0)$. Since $\lim\limits_{(x,y)\to 0} ||(x,y)|| = 0$, the squeeze theorem ensures that $\lim\limits_{(x,y)\to 0} \varepsilon(x,y) = 0$. We conclude $\varepsilon(0,0) = 0$.

ii) The Taylor polynomial $p_2(x,y,z)$ of order 2 around the origin of the function $f(x,y,z)$ of three variables is given by

$$\begin{aligned} p_2(x,y,z) = {} & f(0,0,0) + f'_x(0,0,0)\,x + f'_y(0,0,0)\,y + f'_z(0,0,0)\,z + \\ & \frac{1}{2} f''_{xx}(0,0,0)\,x^2 + \frac{1}{2} f''_{yy}(0,0,0)\,y^2 + \frac{1}{2} f''_{zz}(0,0,0)\,z^2 + \\ & f''_{xy}(0,0,0)\,xy + f''_{xz}(0,0,0)\,xz + f''_{yz}(0,0,0)\,yz \end{aligned}$$

Here we have

$$\begin{aligned} & f(x,y,z) = e^x + y\sinh(z)\,, \\ & f'_x(x,y,z) = e^x\,, && f'_y(x,y,z) = \sinh(z)\,, && f'_z(x,y,z) = y\cosh(z)\,, \\ & f''_{xx}(x,y,z) = e^x\,, && f''_{yy}(x,y,z) = 0\,, && f''_{zz}(x,y,z) = y\sinh(z)\,, \\ & f''_{xy}(x,y,z) = 0\,, && f''_{xz}(x,y,z) = 0\,, && f''_{yz}(x,y,z) = \cosh(z)\,, \end{aligned}$$

where

$$\begin{aligned} & f(0,0,0) = 1, \quad f'_x(0,0,0) = 1, \quad f'_y(0,0,0) = 0, \quad f'_z(0,0,0) = 0, \quad f''_{xx}(0,0,0) = 1, \\ & f''_{yy}(0,0,0) = 0, \quad f''_{zz}(0,0,0) = 0, \quad f''_{xy}(0,0,0) = 0 = f''_{xz}(0,0,0), \quad f''_{yz}(0,0,0) = 1, \end{aligned}$$

and therefore,

$$\begin{aligned} p_2(x,y,z) &= 1 + 1\cdot x + 0\cdot y + 0\cdot z + \frac{1}{2}\cdot 1\cdot x^2 + \frac{1}{2}\cdot 0\cdot y^2 + \frac{1}{2}\cdot 0\cdot z^2 + \\ &\quad\; 0\cdot xy + 0\cdot xz + 1\cdot yz \\ &= 1 + x + \frac{1}{2}x^2 + yz. \end{aligned}$$

iii) We have

$$\begin{aligned} f(x,y) &= \big(\cos(x)\big)^{\frac{1}{2}+\sin(y)} = \exp\Big(\big(\tfrac{1}{2}+\sin(y)\big)\ln\big(\cos(x)\big)\Big)\,, \\ f'_x(x,y) &= -\big(\tfrac{1}{2}+\sin(y)\big)\big(\cos(x)\big)^{\sin(y)-\frac{1}{2}}\sin(x)\,, \\ f'_y(x,y) &= \ln\big(\cos(x)\big)\big(\cos(x)\big)^{\frac{1}{2}+\sin(y)}\cos(y)\,, \end{aligned}$$

where

$$f\Big(\frac{\pi}{3},\frac{\pi}{6}\Big) = \frac{1}{2}\,, \qquad f'_x\Big(\frac{\pi}{3},\frac{\pi}{6}\Big) = -\frac{\sqrt{3}}{2}\,, \qquad f'_y\Big(\frac{\pi}{3},\frac{\pi}{6}\Big) = -\frac{\sqrt{3}}{4}\ln(2)\,.$$

Thus

$$\begin{aligned} p_1(x,y) &= \frac{1}{2} - \frac{\sqrt{3}}{2}\Big(x-\frac{\pi}{3}\Big) - \frac{\sqrt{3}}{4}\ln(2)\Big(y-\frac{\pi}{6}\Big) \\ &= \frac{1}{2} + \frac{\sqrt{3}\pi}{24}(\ln(2)+4) - \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{4}\ln(2)\,y\,. \end{aligned}$$

iv) The partial derivatives of the function $f(x,y,z) = e^{2xz+y}$ are

$$\begin{aligned} &f'_x(x,y,z) = 2z\,e^{2xz+y}, && f'_y(x,y,z) = e^{2xz+y}, && f'_z(x,y,z) = 2x\,e^{2xz+y} \\ &f''_{xx}(x,y,z) = 4z^2\,e^{2xz+y}, && f''_{yy}(x,y,z) = e^{2xz+y}, && f''_{zz}(x,y,z) = 4x^2\,e^{2xz+y} \\ &f''_{xy}(x,y,z) = 2z\,e^{2xz+y}, && f''_{xz}(x,y,z) = (2+4xz)\ ,e^{2xz+y}, && f''_{yz}(x,y,z) = 2x\,e^{2xz+y} \end{aligned}$$

and we have

$$\begin{aligned} &f'_x(0,0,0) = 0, && f'_y(0,0,0) = 1, && f'_z(0,0,0) = 0 \\ &f''_{xx}(0,0,0) = 0, && f''_{yy}(0,0,0) = 1, && f''_{zz}(0,0,0) = 0 \\ &f''_{xy}(0,0,0) = 0, && f''_{xz}(0,0,0) = 2, && f''_{yz}(0,0,0) = 0 \end{aligned}$$

Thus the Taylor polynomial $p_2(x,y,z)$ of order 2 is

$$p_2(x,y,z) = 1 + y + \frac{y^2}{2} + 2xz\,.$$

7. (i)

$$\begin{aligned} f'(t) &= f(t)\cdot\left(\frac{1}{\ln t}\cdot\frac{1}{t}\cdot\sin t + \ln(\ln t)\cdot\cos t\right) \\ &= (\ln t)^{\sin t}\frac{1}{\ln t}\cdot\frac{1}{t}\cdot\sin t + (\ln t)^{\sin t}\ln(\ln t\cdot\cos t \\ &= (\ln t)^{\sin t - 1}\frac{1}{t}\cdot\sin t + (\ln t)^{\sin t}\ln(\ln t)\cdot\cos t. \end{aligned}$$

(ii)

$$\frac{\partial f}{\partial x}(x,y) = e^{-(x\sin y)^2}\frac{\partial(-(x\sin y)^2)}{\partial x} = -2x(\sin y)^2 e^{-(x\sin y)^2}$$

and

$$\frac{\partial f}{\partial y}(x,y) = e^{-(x\sin y)^2}\frac{\partial(-(x\sin y)^2)}{\partial y} = -2x^2 e^{-(x\sin y)^2}\sin y(\sin y)' = -2x^2 e^{-(x\sin y)^2}\sin y\cos y.$$