EXERCISE SHEET 5

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
March 17, 2025

Exercise 1. Determine the equation of the tangent plane to the surface $\{(x,y,z) \in \mathbb{R}^3 : z = x^3y + x^2 + y^2\}$ at the point $(1,1,3) \in \mathbb{R}^3$.

Exercise 2. Determine the equation of the tangent hyperplane to the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 : -2\cos(\pi x) + x^2y + 3e^{xz} + yz = 23\}$$

at the point $(3,2,0) \in \mathbb{R}^3$.

Exercise 3. For $x \in \mathbb{R}$ and $t > 0$ we consider the function $f(x,t)$ defined by

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

(a) Show that f verifies the heat equation, i.e.,

$$\frac{\partial f}{\partial t}(x,t) - \frac{\partial^2 f}{\partial x^2}(x,t) = 0.$$

(b) Let $g(x,y,t)$ be given by $g(x,y,t) = f(x,t)f(y,t)$. Calculate

$$\frac{\partial g}{\partial t}(x,y,t) - \frac{\partial^2 g}{\partial x^2}(x,y,t) - \frac{\partial^2 g}{\partial y^2}(x,y,t).$$

Exercise 4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ by a function defined by

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}\,, & (x,y) \neq (0,0) \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

Calculate $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ and $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$. Is $\dfrac{\partial^2 f}{\partial x \partial y}$ continuous at $(0,0)$?

Exercise 5. Prove that the determinant of Hessian matrix of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = e^x \sin y$ does not depend on the variable y.

Exercise 6. Determine the Taylor polynomial of order n of function f around the given point.

i) $f(x,y) = x^2y + 2xy + 3y^2 - 5x + 1\,,\quad n = 2, \qquad (x_0, y_0) = (0,0).$

ii) $f(x,y,z) = e^x + y\sinh(z)\,,\quad n = 2, \qquad (x_0, y_0, z_0) = (0,0,0).$

iii) $f(x,y) = \big(\cos(x)\big)^{\frac{1}{2}+\sin(y)}\,,\quad n = 1, \qquad (x_0, y_0) = \big(\frac{\pi}{3}, \frac{\pi}{6}\big).$

iv) $f(x,y,z) = e^{2xz+y}\,,\quad n = 2, \qquad (x_0, y_0, z_0) = (0,0,0).$

For i), verify that the error is of order $d^2 \cdot \varepsilon(x,y)$, where $d = \sqrt{x^2+y^2}$.

Exercise 7. Apply the chain rule to calculate the partial derivatives of the functions:

(i) $f :]1,\infty[\to \mathbb{R},\ f(t) = (\ln t)^{\sin t}$.

(ii) $f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = e^{-(x\sin(y))^2}$.