EXERCISE SHEET 4 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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1. f is not continuous at $(0,0)$. One counterexample is

$$f(x,y) = \begin{cases} 0, & \text{if } y = 0 \text{ or } |y| > x^2 \\ 1, & \text{otherwise.} \end{cases}$$

Then f is a well-defined function on $\mathbb{R}^2$ with $f(0,0) = 0$. Let $\alpha, \beta \in \mathbb{R}$. If both are zero, then $f(\alpha t, \beta t) = f(0,0) = 0$. If either $\alpha = 0$ or $\beta = 0$, then also $f(\alpha t, \beta t) = 0$. Finally, if $\alpha, \beta \neq 0$, then we have $f(\alpha t, \beta t) = 0$ for $|t| < \beta/\alpha^2$. Thus, for all $\alpha, \beta \in \mathbb{R}$, we have

$$\lim_{t \to 0} f(\alpha t, \beta t) = 0.$$

However, f is not continuous at $(0,0)$. Let $y = x^2$. Then $f(x, x^2) = 1$ for all $x \neq 0$, thus

$$\lim_{x \to 0} f(x, x^2) = 1.$$

2. (a) If E is empty, then E is closed. If E is not empty, then for all adherent points $\mathbf{x}$ of E and for all sequences $(\mathbf{x}_n)_n$ of elements of E that converges to $\mathbf{x}$: $f(\mathbf{x}_n) = c$ for all n and by continuity of f

$$c = \lim_{n \to \infty} f(\mathbf{x}_n) = f(\mathbf{x})$$

hence $\mathbf{x} \in E$.
(b) For F it is the same idea (replace " $= c$" by " $\leq c$").
(c) The set G is the complementary set of the closed set $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq c\}$, hence it is open.

3. (a)

$$f(x,y) = \begin{cases} \dfrac{x^{2\alpha}}{x^2 + y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

For $(x,y)\neq(0,0)$ the denominator is non-zero and f is a combination of continuous functions. Therefore for all $\alpha>0, f(x,y)$ is continuous $\forall(x,y)\neq(0,0)$. We check the continuity at $(x,y)=(0,0)$. Using polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$ we have

$$\lim_{(x,y)\to(0,0)}\frac{x^{2\alpha}}{x^2+y^2}=\lim_{r\to0}\frac{r^{2\alpha}\cos^{2\alpha}\theta}{r^2}$$

The value of the limit depends on α :

- case $\alpha>1$: The limit is 0 because $\left|r^{2\alpha}\cos^{2\alpha}\theta\right|\leq\left|r^{2\alpha}\right|\to0$
- case $\alpha=1$: The value of the limit is $1\cdot\cos\theta$
- case $0<\alpha<1$: The limit is $+\infty$ if $\cos\theta\neq0$ and the limit is 0 if $\cos\theta=0$.

So f is continuous on $\mathbb{R}^2$ if $\alpha>1$ and is continuous on $\mathbb{R}^2\backslash(0,0)$ when $0<\alpha\leq1$.

(b)

$$f(x,y)=\begin{cases}\dfrac{xy}{(x^2+y^2)^\alpha}, & \text{if } (x,y)\neq0\\ 0 & \text{otherwise.}\end{cases}$$

For $(x,y)\neq(0,0)$ the denominator is non-zero and f is a combination of continuous functions. Therefore for all $\alpha>0, f(x,y)$ is continuous $\forall(x,y)\neq(0,0)$. We check the continuity at $(x,y)=(0,0)$. Using polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$ we have

$$\lim_{(x,y)\to(0,0)}\frac{xy}{(x^2+y^2)^\alpha}=\lim_{r\to0}r^{2(1-\alpha)}\cos\theta\sin\theta$$

The value of the limit depends on α :

- $\alpha=1$: the limit is $\cos\theta\sin\theta$.
- $0<\alpha<1$: the limit is 0.
- $\alpha>1$: depending on θ it can be $0,+\infty$ and $-\infty$.

So f is continuous on $\mathbb{R}^2$ if $0<\alpha<1$ and is continuous on $\mathbb{R}^2\backslash(0,0)$ when $\alpha\geq1$.

4.

$$\begin{aligned}&\lim_{(x,y)\to(0,0)}\frac{3\left(x^2+y^2\right)}{\sqrt{x^2+y^2+4}-2}\\ &=\lim_{(x,y)\to(0,0)}\frac{3\left(x^2+y^2\right)\left(\sqrt{x^2+y^2+4}+2\right)}{x^2+y^2}\\ &=\lim_{(x,y)\to(0,0)}3\left(\sqrt{x^2+y^2+4}+2\right)=12.\end{aligned}$$

5. The function is partially differentiable since polynomials and trigonometric functions are differentiable.

$$\nabla f(x,y) = \begin{pmatrix} 2x + y\cos x - 2y^2 \sin x \cos x \\ \sin x + 2y\cos^2 x \end{pmatrix}$$

Note that $f(0,1) = 1$ and

$$\nabla f(0,1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

6. Let $r = \sqrt{x^2+y^2}, r \geq 0. f$ is differentiable at $(0,0)$ and $d_0 f(x,y) = 0$ since

$$\lim_{r\to 0_+} \frac{f(x,y) - f(0,0)}{r} = \lim_{r\to 0} r \sin\left(\frac{1}{r}\right) = 0$$

If $(x,y) \neq (0,0)$ the function f is partially differentiable (even differentiable) and noting that f is radially symmetric:

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= \frac{x}{r}\left(2r\sin r^{-1} - \cos r^{-1}\right) \\ \frac{\partial f(x,y)}{\partial y} &= \frac{y}{r}\left(2r\sin r^{-1} - \cos r^{-1}\right) \end{aligned}$$

These functions don't have any limits when $(x,y) \to (0,0)$ (because of $\cos r^{-1}$).

7. On the open set $\mathbb{R}^2 \backslash \{(0,0)\}$, f admits partial derivatives (cf Analysis I). They are continuous because written using the continuous functions $(x,y) \to x$ and $(x,y) \to y$ and operations that preserve continuity. In our case, the only possible problematic point is therefore $(0,0)$.

(i) The function is continuous at any point of $\mathbb{R}^2 \backslash \{(0,0)\}$ (the composition of continuous functions gives continuous functions). For $(x,y) \neq (0,0)$,

$$0 \leq |f(x,y)| \leq \frac{||(x,y)||^3}{x^2+y^2} = ||(x,y)||.$$

But $\lim\limits_{(x,y)\to(0,0)} ||(x,y)|| = 0$ and therefore, by the squeeze theorem $\lim\limits_{(x,y)\to(0,0)} |f(x,y)| = 0$. From where $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. As $f(0,0) = 0$, f is also continuous at $(0,0)$.

(ii) The functions $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are well-defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. At $(0,0)$ we have:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0\,,$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0\,.$$

(iii) The function f has continuous partial derivatives on the open set $\mathbb{R}^2\setminus\{(0,0)\}$, hence, f is differentiable for $(x,y)\neq 0$.

(iv) Suppose that f is differentiable at $(x,y)=(0,0)$, then

$$f(x,y)=f(0+x,0+y)=f(0,0)+\frac{\partial f}{\partial x}(0,0)\ x+\frac{\partial f}{\partial y}(0,0)\ y+r(x,y)$$

with $\lim\limits_{(x,y)\to(0,0)}\frac{r(x,y)}{||(x,y)-(0,0)||}=0$. Since $f(0,0)=\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$, we have that

$$\frac{r(x,y)}{||(x,y)||}=\frac{f(x,y)}{\sqrt{x^2+y^2}}=\frac{x^2y}{(x^2+y^2)\sqrt{x^2+y^2}}\ .$$

However,

$$\lim_{x\to 0+}\frac{r(x,x)}{||(x,x)||}=\lim_{x\to 0+}\frac{x^3}{2\sqrt{2}x^3}=\frac{1}{2\sqrt{2}}\neq 0\,,$$

contradicts our starting assumption.

(v) The functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ cannot both be continuous in $(0,0)$ because otherwise the function f would be differentiable at $(0,0)$, which is not the case (cf. (iv)).

To see this directly, consider for instance

$$\frac{\partial f}{\partial x}(x,y)=\frac{2xy}{x^2+y^2}-\frac{2x^3y}{(x^2+y^2)^2}$$

and

$$\lim_{x\to 0}\frac{\partial f}{\partial x}(x,x)=\frac{1}{2}\neq 0=\frac{\partial f}{\partial x}(0,0)\,.$$

8. Let

$$f(x,y)=\begin{cases} x-y+\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\,,\\ 0 & \text{if } (x,y)=(0,0)\,,\end{cases}$$

whose partial derivatives at $(0,0)$ are:

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{x-0+\frac{x\cdot 0}{x^2+0^2}-0}{x}=\lim_{x\to 0}1=1\,,$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{0-y+\frac{0\cdot y}{0^2+y^2}-0}{y}=\lim_{y\to 0}(-1)=-1\,.$$

The function f is not differentiable at $(0,0)$ because it is not even continues in $(0,0)$. In particular

$$\lim_{x\to 0}f(x,x)=\lim_{x\to 0}\left(x-x+\frac{x\cdot x}{x^2+x^2}\right)=\lim_{x\to 0}\frac{1}{2}=\frac{1}{2}\neq f(0,0).$$

In fact the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't even exist because

$$\lim_{x\to 0} f(x,x) = \frac{1}{2} \neq 0 = \lim_{x\to 0} f(x,0).$$