EXERCISE SHEET 4

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
March 10, 2025

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with $f(0,0) = 0$ such that for all $\alpha \in \mathbb{R}$ and all $\beta \in \mathbb{R}$ we have

$$\lim_{t \to 0} f(\alpha t, \beta t) = 0.$$

Is f is continuous at $(0,0)$? If yes, prove it, otherwise construct a counterexample.

Exercise 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that for all $c \in \mathbb{R}$:

(a) $E = \{\mathbf{x} \in X : f(\mathbf{x}) = c\}$ is closed.
(b) $F = \{\mathbf{x} \in X : f(\mathrm{x}) \leq c\}$ is closed.
(c) $G = \{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ is open.

Exercise 3. Study continuity of following functions as a function of $\alpha > 0$.

(a)

$$f(x,y) = \begin{cases} \dfrac{x^{2\alpha}}{x^2 + y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2 + y^2)^{\alpha}}, & \text{if } (x,y) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: use polar coordinates)

Exercise 4. Find:

$$\lim_{(x,y) \to (0,0)} \frac{3\left(x^2 + y^2\right)}{\sqrt{x^2 + y^2 + 4} - 2}$$

if it exists.

Exercise 5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function given by

$$f(x,y) = x^2 + y \sin x + y^2 \cos^2 x.$$

Show that f is partially differentiable and give the gradient of f.

Exercise 6. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$.

Exercise 7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}\,, & (x,y) \neq (0,0) \\ 0\,, & (x,y) = (0,0). \end{cases}$$

(i) Show that f is continuous on $\mathbb{R}^2$.
(ii) Show that functions $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are defined on $\mathbb{R}^2$.
(iii) Show that f is differentiable for all $(x,y) \neq (0,0)$.
(iv) Show that f is not differentiable at $(x,y) = (0,0)$.
(v) Are the functions $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ continuous at $(x,y) = (0,0)$?

Exercise 8. Give an example of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial f}{\partial x}(0,0) = 1$ and $\frac{\partial f}{\partial y}(0,0) = -1$, but is not differentiable at $(0,0)$.