EXERCISE SHEET 3 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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1. (a) $f(x,y) = 16 - x^2 - y^2$, $p = (2\sqrt{2}, \sqrt{2})$: $f(2\sqrt{2}, \sqrt{2}) = 6$, so the level curve has the equation $x^2 + y^2 = 10$.

(b) $f(x,y) = \sqrt{x^2 - 1}$, $p = (1,0)$: $f(1,0) = 0$, so the level curve has the equation $x^2 = 1$ which consists of two lines $x = 1$ and $x = -1$.

(c) $f(x,y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}$, $p = (0,1)$: We have that

$$\int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \arcsin y - \arcsin x.$$

where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. We also have that $f(0,1) = \pi/2$. In order for $\arcsin y - \arcsin x$ to be equal to $\pi/2$ we must have $0 \leq \arcsin y \leq \pi/2$ and $-\pi/2 \leq \arcsin x \leq 0$ meaning $0 \leq y \leq 1$ and $-1 \leq x \leq 0$. So the equation of the curve is given by

$$\arcsin y - \arcsin x = \frac{\pi}{2} \Longrightarrow y = \sin\left(\frac{\pi}{2} + \arcsin x\right) \Longrightarrow y = \sqrt{1 - x^2}, \quad x \leq 0$$

2. (i) We have $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^2 - 3y}{x + 2y^2} = \frac{4-3}{2+2} = \frac{1}{4}$.

(ii) Note that

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} \right| \leq |xy| \leq x^2 + y^2.$$

Hence

$$\lim_{(x,y)\to(0,0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0.$$

(iii) For $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with $x \neq 0$ we have

$$|f(x,y)| = \left| \frac{x^2 y}{x^2 + y^4} \right| \leq \left| \frac{x^2 y}{x^2 + 0} \right| \leq |y|.$$

For $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with $x = 0$ we have $f(x,y) = 0$, therefore, in any case we have $|f(x,y)| \leq |y|$. Now, let (x_n, y_n) be a sequence such that $(x_n, y_n) \neq (0,0)$, yet, $\lim_{n\to\infty}(x_n, y_n) = (0,0)$. Hence,

$\lim_{n\to\infty} y_n = 0$ and thus $\lim_{n\to\infty} |f(x_n, y_n)| \leq \lim_{n\to\infty} |y_n| = 0$. In conclusion, $\lim\limits_{(x,y)\to(0,0)} |f(x,y)| = 0$.

3. (i) For a sequence of the form $(x_n, 0)$, with $x_n \neq 0$ and $\lim_{n\to\infty} x_n = 0$ we have $\lim_{n\to\infty} f(x_n, 0) = \dfrac{x_n \cdot 0^2}{x_n^2 + 0^4} = \lim_{n\to\infty} 0 = 0$. However, for a sequence of the form (y_n^2, y_n) with $y_n \neq 0$ and $\lim_{n\to\infty} y_n = 0$ we have $\lim_{n\to\infty} f(y_n^2, y_n) = \lim_{n\to\infty} \dfrac{y_n^2 y_n^2}{(y_n^2)^2 + y_n^4} = \dfrac{1}{2}$. As such, the limit does not exist.

(ii) Consider the following two limits for f:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{5x^2} = 0 \qquad \text{and} \qquad \lim_{x\to 0} f(x,2x) = \lim_{x\to 0} \frac{2x^2}{x^2} = 2\,.$$

Hence, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(iii) Let

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

The limit as $(x,y) \to (0,0)$ doesn't exist since $f(0,y) = -1$ for $y \neq 0$ and $f(x,0) = 1$ if $x \neq 0$.

(iv) We have

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0} t^2 \frac{0}{4t^4} = 0\,,$$

However, we also have

$$\lim_{t\to 0} f(2t,t) = \lim_{t\to 0} 2t^2 \frac{4t^2 - t^2}{(4t^2 + t^2)^2} = \lim_{t\to 0} \frac{6t^4}{25t^4} = \frac{6}{25} \neq 0\,.$$

As these limits do not agree, the limit for $(x,y) \to (0,0)$ fails to exist.

4.

(i) We use polar coordinates: $\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \end{cases}$. Hence, $x^2 + y^2 = r^2$, therefore

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1.$$

(ii) Use again polar coordinates: $\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \end{cases}$. Then,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{1-\cos(r)}{r^2} = \lim_{r\to 0} \frac{1-\cos(r)^2}{r^2(1+\cos(r))} = \lim_{r\to 0} \left(\frac{\sin(r)}{r}\right)^2 \cdot \frac{1}{1+\cos(r)} = \frac{1}{2}.$$

(iii) Using polar coordinates $\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \end{cases}$, we have

$$3x^3 - 2y^3 = r^3\left(3\cos(\varphi)^3 - 2\sin(\varphi)^3\right) \qquad \text{and} \qquad x^2 + y^2 = r^2$$

and thus

$$f(x,y) = r\left(3\cos(\varphi)^3 - 2\sin(\varphi)^3\right),$$

such that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} r\left(3\cos(\varphi)^3 - 2\sin(\varphi)^3\right) = 0.$$

(iv) We use polar coordinates. We substitute $x = r\cos\theta$ and $y = r\sin\theta$ and investigate the limit of resulting expression as $r \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{x^2+y^2} = \lim_{r\to 0} \frac{e^{-\frac{1}{r}}}{r^2} = 0.$$