EXERCISE SHEET 2 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
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1. By the properties of the scalar product, we get

$$\begin{aligned}\|\mathbf{x}\pm\mathbf{y}\|_2^2 &= \langle \mathbf{x}\pm\mathbf{y}, \mathbf{x}\pm\mathbf{y}\rangle \\ &= \langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle \pm \langle \mathbf{x},\mathbf{y}\rangle \pm \langle \mathbf{y},\mathbf{x}\rangle \\ &= \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 \pm \langle \mathbf{x},\mathbf{y}\rangle \pm \langle \mathbf{y},\mathbf{x}\rangle,\end{aligned}$$

and hence

$$\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 - 2\|\mathbf{x}\|_2^2 - 2\|\mathbf{y}\|_2^2 = 0.$$

2. For all $\mathbf{x}, \mathbf{y} \in E$ and real number λ, we have:

$$0 \leq \langle \mathbf{x}-\lambda\mathbf{y}, \mathbf{x}-\lambda\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{x}\rangle - 2\lambda\langle \mathbf{x},\mathbf{y}\rangle + \lambda^2\langle \mathbf{y},\mathbf{y}\rangle.$$

We minimize the above equation with regard to λ: If $\mathbf{y} = \mathbf{0}$, there is nothing to prove (both two sides of Cauchy-Schwarz's inequality are equal to zero). If $\mathbf{y} \neq \mathbf{0}$, then we have $\langle \mathbf{y},\mathbf{y}\rangle > 0$ by the positivity of the scalar product, and the minimum of this polynomial of degree 2 with regard to λ is obtained when $\lambda = \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle}$, with which we get

$$0 \leq \langle \mathbf{x},\mathbf{x}\rangle - \frac{\langle \mathbf{x},\mathbf{y}\rangle^2}{\langle \mathbf{y},\mathbf{y}\rangle}$$

hence the proof for the Cauchy-Schwarz's inequality is finished.

3. (a) Note first that we can assume $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ since otherwise the inequality is trivial (the two members are equal to zero). By the triangle inequality for the absolute value, we derive the basic inequality:

$$|\langle \mathbf{x},\mathbf{y}\rangle| = \left|\sum_{k=1}^{n} x_k y_k\right| \leq \sum_{k=1}^{n} |x_k|\cdot|y_k| \tag{1}$$

From this it follows that $|\langle \mathbf{x}, \mathbf{y}\rangle| \le \|\mathbf{x}\|_1 \|\mathbf{y}\|_\infty$, showing Hölder's inequality for $p = 1, q = \infty$.

Let $p > 1$. By Young's inequality, for all $t > 0$ and all k, we have

$$|x_k| \cdot |y_k| = |tx_k| \cdot \left|t^{-1} y_k\right| \le \frac{t^p\,|x_k|^p}{p} + \frac{t^{-q}\,|y_k|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

By summing the above and using (1), we obtain

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \le \frac{t^p \|\mathbf{x}\|_p^p}{p} + \frac{t^{-q}\|\mathbf{y}\|_q^q}{q}$$

for all $t > 0$. Then choose

$$t = \left(\frac{\|\mathbf{y}\|_q^q}{\|\mathbf{x}\|_p^p}\right)^{\frac{1}{p+q}} \tag{2}$$

in the above, and then we have

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \le \left(\frac{1}{p} + \frac{1}{q}\right)\left(\|\mathbf{x}\|_p^{\frac{pq}{p+q}} \|\mathbf{y}\|_q^{\frac{pq}{p+q}}\right) = \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

Comment: In case that one cannot see that picking the value of t in (2) will give the result, there is another way to continue the solution. Consider the function

$$f(t) = \frac{t^p \|\mathbf{x}\|_p^p}{p} + \frac{t^{-q}\|\mathbf{y}\|_q^q}{q}, \quad t > 0,$$

and by studying its monotonicity (though its derivative), one can see that f has a global maximum at the point t_0 given by (2).

(b) **Non-negativity**: $\|\mathbf{x}\|_\infty = \max_{1\le k\le n} |x_k| = 0$ if and only if $|x_k| = 0$ for all k which is equivalent to $\mathbf{x} = \mathbf{0}$.

Homogeneity: For all $\lambda \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$, using the homogeneity of the absolute value, we have

$$\|\lambda \mathbf{x}\|_\infty = \max_{1\le k\le n} |\lambda x_k| = \max_{1\le k\le n} |\lambda|\,|x_k| = |\lambda| \max_{1\le k\le n} |x_k| = |\lambda| \|\mathbf{x}\|_\infty.$$

Triangle inequality: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, using the triangle inequality for the absolute value, we have

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|_\infty &= \max_{1\le k\le n} |x_k + y_k| \le \max_{1\le k\le n} |x_k| + |y_k| \\ &\le \max_{1\le k\le n} |x_k| + \max_{1\le k\le n} |y_k| \\ &= \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.\end{aligned}$$

(c) **Non-negativity**: $\|\mathbf{x}\|_1 = \sum_{k=1}^n |x_k| = 0$ if and only if $|x_k| = 0$ for all k which is equivalent to $\mathbf{x} = \mathbf{0}$.

Homogeneity: For all $\lambda \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$, using the homogeneity of the absolute value, we have

$$\|\lambda \mathbf{x}\|_1 = \sum_{k=1}^{n} |\lambda x_k| = |\lambda| \sum_{k=1}^{n} |x_k| = |\lambda| \|\mathbf{x}\|_1.$$

Triangle inequality: For all $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$, using the triangle inequality for the absolute value, we have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_1 &= \sum_{k=1}^{n} |x_k + y_k| \leq \sum_{k=1}^{n} |x_k| + |y_k| \\ &= \|\mathbf{x}\| \,\|_1 + \| \,\mathbf{y}\|_1.\end{aligned}$$

4. (a) $S = \{(x,y) \in \mathbb{R}^2 \colon 0 \leq y < (1+x^2)e^{-|x|}\}$ is neither open nor closed:

- For any $x \in \mathbb{R}$, we have that the point $(x,0)$ is not in the interior of S. To see this, for any $\varepsilon > 0$, we have that $(x - \varepsilon/2, -\varepsilon/2) \notin S$, hence $B((x,0),\varepsilon) \not\subset S$.
- One way to see that S is not closed is by proving that the point $(0,1)$, which clearly does not belong in S, is a boundary point of S. One quick way to show this is by considering the sequence $((0, 1-\frac{1}{n}))_{n\in\mathbb{N}}$ which converges to $(0,1)$.

The set $T = \{(x,y) \in \mathbb{R}^2 \colon 1 < x^2 + 4y^2 < 4\}$ is open but not closed:

- For any $(a,b) \in T$, the open ball $B((a,b),r)$ with $2r < \min(2 - \sqrt{a^2+4b^2}, \sqrt{a^2+4b^2} - 1)$ is contained in T.
- To show that it is not closed just argue that the point $(0,1)$ is a boundary point that is not in T.

$\mathbb{Q}$ is neither open nor closed: By a result from the course Analyse I, the set $\mathbb{Q}$ is dense in $\mathbb{R}$. Between two real numbers there always exists a rational number and vice versa (see also exercises Analyse I, chapter 1). Hence, all point of $\mathbb{Q}$ is a boundary point. So

$$\overset{\circ}{\mathbb{Q}} = \emptyset, \quad \partial\mathbb{Q} = \overline{\mathbb{Q}} = \mathbb{R}.$$

(b) $\overset{\circ}{S} = \{(x,y) \in \mathbb{R}^2 \colon 0 < y < (1+x^2)e^{-|x|}\}$: In (a), we saw that $\mathbb{R} \times \{0\}$ is disjoint from the interior of S, hence it suffices to show that any point (x,y) with $0 < y < (1+x^2)e^{-|x|}$ belongs in the interior of S. Let $f(x) = (1+x^2)e^{-|x|},\ x \in \mathbb{R}$. Let (a,b) with $0 < b < f(a)$. Then:

- there exists $\delta_1 > 0$ such that $]b - \delta_1, b + \delta_1[\subset]0, f(a)[$, and
- by continuity of f, there exists $\delta_2 > 0$ such that for any $x \in]a - \delta_2, a + \delta_2[$, we have $f(x) > b + \delta_1$.

Therefore, by picking $\varepsilon = \min(\delta_1, \delta_2) > 0$, we have that

$$B((a,b),\varepsilon) \subset \{(x,y) \in \mathbb{R}^2 \colon 0 < y < (1+x^2)e^{-|x|}\}.$$

$\partial S = \{(x,y) \in \mathbb{R}^2 \colon y = 0 \text{ or } y = (1+x^2)e^{-|x|}\} = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times f(\mathbb{R}))$: It is not hard to check that any point (x,y) with $y = 0$ or $y = f(x)$ is a boundary point and all other points are not (either using open balls or using convergence of sequences).

Finally, $\overline{S} = S \cup \partial S = \{(x,y) \in \mathbb{R}^2 \colon 0 \leq y \leq (1+x^2)e^{-|x|}\}$.

Similarly, for T we have:

$$\overset{\circ}{T} = T$$

$$\partial T = \{(x,y) \in \mathbb{R}^2 \colon x^2 + 4y^2 = 1 \text{ or} x^2 + 4y^2 = 4\}$$

$$\overline{T} = \{(x,y) \in \mathbb{R}^2 \colon 1 \leq x^2 + 4y^2 \leq 4\}.$$

The interior, boundary and closure of $\mathbb{Q}$ are calculated in (a).

(c)

$$\text{Area}(S) = \int_{-\infty}^{\infty} \left(1 + x^2\right) e^{-|x|} dx = 2\int_0^{\infty} \left(1 + x^2\right) e^{-|x|} dx = 2\Gamma(1) + 2\Gamma(3) = 6.$$

The boundary of T is given by the two ellipses $E(1, 1/2)$ and $E(2, 1)$. Note that $E(1, 1/2) \subset E(2, 1)$. So

$$\text{Area}(T) = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}.$$

5. (a) The sequence $((0, 3 + \frac{1}{n}, 0))_{n\in\mathbb{N}}$ belongs in the set C, but its limit $(0, 3, 0)$ does not.

(b) C is not open: The point $(0, 5, 0) \in C$ and for any $\varepsilon > 0$ the open ball with center $(0, 5, 0)$ and radius is not contained in C.