EXERCISE SHEET 2

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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Exercise 1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\|\cdot\|_2$ be the Euclidean norm, i.e.

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Calculate

$$\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 - 2\|\mathbf{x}\|_2^2 - 2\|\mathbf{y}\|_2^2.$$

Exercise 2. (Cauchy-Schwarz's inequality in the Euclidean space). Let $(E, \langle \cdot, \cdot \rangle)$ be a Euclidean space. Show that for all $\mathbf{x}, \mathbf{y} \in E$:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \cdot \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Exercise 3. (Hölder's inequality and norms on $\mathbb{R}^{\mathbf{n}}$). For $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ is a real number, let

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}.$$

Moreover, let

$$\|\mathbf{x}\|_\infty = \max_{1 \leq k \leq n} |x_k|.$$

(a) Show Hölder's inequality: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\frac{1}{p} + \frac{1}{q} = 1$ (with the convention that if $p = 1$, then $q = \infty$ and vice versa):

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

(Hint: use Young's inequality: for p and q satisfying $1/p + 1/q = 1$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{for all } a, b \in \mathbb{R}$$

to show that for all $t > 0$:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \frac{t^p \|\mathbf{x}\|_p^p}{p} + \frac{t^{-q} \|\mathbf{y}\|_q^q}{q}$$

and deduce Hölder's inequality from it.)

(b) Show that $\|\mathbf{x}\|_\infty$ defines a norm on $\mathbb{R}^n$.

(c) Show that $\|\mathbf{x}\|_1$ defines a norm on $\mathbb{R}^n$.

Comment: To show that the above "define a norm", you need to show that they satisfy the properties of **non-negativity**, **homogeneity** and the **triangle inequality** (please find non-negativity etc. on page 8 in the lecture notes).

Exercise 4. Let $S = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq y < \left(1 + x^2\right) e^{-|x|}\right\}$, $T = \left\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + 4y^2 < 4\right\}$ and consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$.

(a) Determine whether these sets are open or closed.

(b) Calculate the interior, boundary and closure of these sets.

(c) Calculate the area of S and T.

Hint: Useful formulas:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \Gamma(n+1) = n!$$

$$\int_0^x \sqrt{1 - t^2} dt = \frac{x\sqrt{1 - x^2} + \arcsin x}{2}$$

$$E_{a,b} = \left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}, \ a, b > 0, \quad \text{Area}\left(E_{a,b}\right) = \pi ab.$$

Exercise 5. Let

$$C = \{(x, y)\colon 9 < x^2 + y^2 \leq 25\} \times [0, 1]$$

(a) Use convergence of sequences to determine whether C is closed.

(b) Is C open?

(c) Draw the set C.