EXERCISE SHEET 1 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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1. (a) True. This is the statement of l'Hopital's rule.

(b) False. Take, for example, $f(x) = x + \sin x$ and $g(x) = x$. We have $\frac{f'(x)}{g'(x)} = 1 + \cos x$ which has no limit at infinity. However, $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \left(1 + \frac{\sin x}{x}\right) = 1$.

(c) False. Take the counterexample for the preceding question.

(d) True. It follows by applying the mean value theorem to the function

$$h(z) = f(z) - f(x) - \frac{f(y) - f(x)}{g(y) - g(x)}(g(z) - g(x)), \quad z \in [x, y].$$

(e) True. Since g is differentiable on $\mathbb{R}$, the function $\sin g(x)$ is differentiable on $\mathbb{R}$. If $g(a) = 0$, then $\sin g(a) = 0$ and we may apply l'Hopital's rule which guarantees the existence of the limit. If $g(a) \neq 0$, then by continuity, the limit is simply $\frac{\sin g(a)}{g(a)}$.

(f) False. Take, for example, $g(x) = x$ and $a = 1$. We have $g(a) = 1 \neq 0$ (note that in this case l'Hopital's rule does not apply). By continuity, the limit is $\frac{\sinh 1}{1} \neq \cosh 1$.

2. (a) False. Since 7 is odd, f admits an inflexion point at a.

(b) True. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$ be the Taylor expansion of f to order n at 0. We then have $f(-x) = a_0 - a_1 x + \cdots + (-1)^n a_n x^n + o(x^n)$. Since f is odd and the Taylor expansion is unique, we conclude that $a_{2m} = 0$ for all $0 \leq 2m \leq n$. As $a_k = \frac{f^{(k)}(0)}{k!}$, we obtain the result.

(c) True. This was shown in class.

3. (a)

$$\lim_{x\to 0} \frac{\sin x}{\ln\left(\frac{1}{1+x}\right)} = \lim_{x\to 0} \frac{\sin x}{-\ln(1+x)} = \lim_{x\to 0} \frac{\cos x}{-\frac{1}{1+x}} = -1.$$

(b) For $\alpha > 0$ using l'Hopital's rule

$$\lim_{x\to 0^+} x^\alpha \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^\alpha}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\alpha x^{\alpha-1}}{x^{2\alpha}}} = \lim_{x\to 0^+} -\frac{1}{\alpha} x^\alpha = 0.$$

(c) By a (long and) direct calculation using l'Hopital's rule

$$\lim_{x\to 1} \frac{\arctan\left(\frac{1-x}{1+x}\right)}{x-1} = \lim_{x\to 1} \frac{-\frac{1}{1+x^2}}{1} = -\frac{1}{2}.$$

Alternatively, we write $y = \frac{1-x}{1+x}$, so $x - 1 = \frac{-2y}{1+y}$ and

$$\lim_{x\to 1} \frac{\arctan\left(\frac{1-x}{1+x}\right)}{x-1} = \lim_{y\to 0} \frac{-(y+1)\arctan y}{2y} = \lim_{y\to 0} \frac{-\arctan y - \frac{y+1}{1+y^2}}{2} = -\frac{1}{2}.$$

(d) Using truncated Taylor expansions we have that $\sinh x = x + O\left(x^3\right)$ and $\ln(1+y) = y + O\left(y^2\right)$, and so

$$\lim_{x\to 0} \frac{\ln\left(1+x^2\right)}{\sinh^2 x} = 1.$$

(e) Since $\ln((x-1)^2) \to -\infty$ as $x \to 1$ and the functions of the numerator are bounded, it follows that

$$\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)\sin(x-1)}{\ln\left((x-1)^2\right)} = 0.$$

(f) Since $e^{\frac{1}{x}}$ goes to infinity as $x \to 0^+$ and all the other terms are bounded, we have that

$$\lim_{x\to 0+} \frac{\cos x - \cos\frac{1}{x}}{e^x - e^{\frac{1}{x}}} = 0.$$

4. (a) $f(x) = xe^{-x^2}$ is of class C^∞ with

$$f'(x) = \left(1-2x^2\right)e^{-x^2} \quad f''(x) = -2x\left(3-2x^2\right)e^{-x^2}.$$

Stationary points: $x_1 = -\frac{\sqrt{2}}{2}$ (strict local minimum, $f(x_1) = -\frac{\sqrt{2}}{2}e^{-\frac{1}{2}}$) and $x_2 = \frac{\sqrt{2}}{2}$ (strict local maximum, $f(x_2) = \frac{\sqrt{2}}{2}e^{-\frac{1}{2}}$). These are in fact global extrema since

$$\lim_{x\to\pm\infty} f(x) = 0.$$

Inflexion points: $x = 0$ and $x = \pm\frac{\sqrt{6}}{2}$.

(b) On the boundary $f(-1) = f(3) = 0$ and

$$\lim_{x\to\pm\infty} f(x) = \infty$$

More precisely, one observes that

$$\lim_{n\to-\infty} f(x) - (1-x) = 0 \quad \lim_{n\to\infty} f(x) - (x-1) = 0$$

so that there are asymptotes $y = 1 - x$ when $x \to -\infty$ and $y = x - 1$ when $x \to +\infty$.

f is of class C^∞ on the open set $]-\infty,-1]\cup[3,\infty[$ with

$$f'(x)=\frac{x-1}{\sqrt{(x+1)(x-3)}}\qquad f''(x)=\frac{-4}{(\sqrt{(x+1)(x-3)})^3}.$$

f is strictly monotonic and strictly concave on its domain.

5. (a) We note that $f(x)=\exp(x\ln x-x)$ and so $f'(x)=(\ln x)f(x)$. Hence $x=1$ is the unique stationary point of the function f as $f(x)>0$ and $\ln x$ is strictly increasing.

(b) f has a strict local minimum at $x=1$ since $f'(x)<0$ if $x<1$ and $f'(x)>0$ if $x>1$. The truncated Taylor expansion of order 4 at this point is given by

$$f(x)=e^{-1}\left(1+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3+\frac{5}{24}(x-1)^4\right).$$

(c) Note that we can rewrite a^b as $e^{b\ln a}$ using this trick we can write $f(x)$ as

$$f(x)=e^{x\ln x-x}$$

Since $g(x)=e^x$ is a continuous function, we have

$$\lim_{x\to0^+}e^{x\ln x-x}=e^{\lim_{x\to0^+}(x\ln x-x)}$$

We saw in **Exercise 3** (b) that $\lim_{x\to0^+}x\ln x=0$ so

$$\lim_{x\to0^+}f(x)=e^0=1$$

By l'Hopital's rule (one may verify the required conditions are satisfied) we get

$$\lim_{x\to0_+}\frac{f(x)-1}{x\ln x}=\lim_{x\to0_+}\frac{f'(x)}{1+\ln x}=\lim_{x\to0_+}\frac{(\ln x)f(x)}{1+\ln x}=\left(\lim_{x\to0_+}\frac{\ln x}{1+\ln x}\right)\left(\lim_{x\to0_+}f(x)\right)$$
$$=\left(\lim_{x\to0_+}\frac{\frac{1}{x}}{x}\right)\left(\lim_{x\to0_+}f(x)\right)=1.$$

6. (a) By the change of variable $s=\sqrt{t}$, i.e., $t=s^2$,

$$\int_0^{+\infty}e^{-\sqrt{t}}dt=\int_0^{+\infty}2se^{-s}ds=2\Gamma(2)=2,$$

where Γ denotes the Gamma function.

(b)

$$\int_1^{+\infty}\frac{\ln t}{t^3}dt=\frac{1}{2}\int_1^{+\infty}\ln t\left(-t^{-2}\right)'dt=\frac{1}{2}\int_1^{+\infty}\frac{1}{t^3}dt=\frac{1}{4}.$$

(c)

$$\int_0^{+\infty} \frac{\arctan t}{1+t^2} dt = \left. \frac{\arctan^2 t}{2} \right|_0^{\infty} = \frac{\pi^2}{8}.$$

7. (a) The integral converges if $\alpha < 1$ and diverges otherwise.

(b) The integral converges if $\alpha > 1$ and diverges otherwise.