EXERCISE SHEET 14 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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1.

i) $y(x) = C_1 \cos(x) + C_2 \sin(x)$

ii) $y(x) = C_1 \cosh(x) + C_2 \sinh(x)$ or $y(x) = C_1 e^x + C_2 e^{-x}$

iii) $y(x) = Ce^{\lambda x}$

For $i)$ and $ii)$, we can use the characteristic equation. That of $i)$ is $\lambda^2 + 1 = 0$, which admits the conjugate complex roots $\lambda_{1,2} = \pm i$. That of $ii)$ is $\lambda^2 - 1 = 0$, which admits the real roots $\lambda_{1,2} = \pm 1$. Without going through the characteristic equation, we can also see that the functions $\cos(x), \sin(x)$ for $iii)$ and $\cosh(x), \sinh(x)$ and $e^{\pm x}$ for $iv)$ are linearly independent solutions of the given equation.

For $iii)$, it is easy to see that $y(x) = Ce^{\lambda x}$ satisfies $y'(x) = \lambda y(x)$. We can also use the integrating factor method for example:

$$\begin{aligned} y' = \lambda y \;\Leftrightarrow\; y' - \lambda y = 0 \;\Leftrightarrow\; e^{-\lambda x} y' - e^{-\lambda x}\lambda y = 0 \\ \Leftrightarrow\; (e^{-\lambda x} y)' = 0 \;\Leftrightarrow\; e^{-\lambda x} y = C \;\Leftrightarrow\; y(x) = Ce^{\lambda x} \end{aligned}$$

2. To solve these homogeneous equations we use the method with the characteristic equation.

i) The characteristic equation $3\lambda^2 - 4\lambda + 1 = 0$ admits the real roots $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{3}$, from where one obtains the general solution

$$y(x) = C_1 e^x + C_2 e^{\frac{1}{3}x}\,, \qquad x, C_1, C_2 \in \mathbb{R}\ .$$

ii) The characteristic equation $3\lambda^2 - 4\lambda + 2 = 0$ admits conjugate complex roots $\lambda_{1,2} = \frac{2}{3} \pm \frac{\sqrt{2}}{3} i$, hence the general solution becomes

$$y(x) = e^{\frac{2}{3}x}\left(C_1 \cos\!\left(\frac{\sqrt{2}x}{3}\right) + C_2 \sin\!\left(\frac{\sqrt{2}x}{3}\right)\right)\,, \qquad x, C_1, C_2 \in \mathbb{R}\ .$$

iii) The characteristic equation $3\lambda^2 - 4\lambda + \frac{4}{3} = 0$ admits the double root $\lambda = \lambda_1 = \lambda_2 = \frac{2}{3}$. Thus one term of the general solution is $y_1(x) = e^{\frac{2}{3}x}$ but a second term must be found (as it is a second order equation). In the case of a double root for a second-order linear equation, a second linearly independent solution is obtained by multiplying $e^{\lambda_1 x}$ by x: $y_2(x) = xe^{\lambda_1 x}$. Hence the general solution

$$y(x) = (C_1 + C_2 x)\, e^{\frac{2}{3}x}\,, \qquad x, C_1, C_2 \in \mathbb{R}\ .$$

3.

i) The characteristic equation of this equation is $\lambda^2+2\lambda-3=0$ which admits the roots $\lambda_1=1$, $\lambda_2=-3$.

Thus the general solution of the given equation is

$$y_{\text{hom}}(x)=C_1e^{x}+C_2e^{-3x}\ .$$

ii) To find a particular solution of the inhomogeneous DE, we set

$$y_{\text{part}}=C_1(x)\,e^{x}+C_2(x)\,e^{-3x}\,.$$

Reasoning as in exercise 1 above, we see that the functions C_1 and C_2 satisfy the system

$$\begin{cases} {C_1}'(x)e^{x}+{C_2}'(x)e^{-3x}=0\\ {C_1}'(x)e^{x}-3{C_2}'(x)e^{-3x}=5\sin(3x)\end{cases}$$

which has the solution

$$\begin{pmatrix}{C_1}'(x)\\ {C_2}'(x)\end{pmatrix}=\begin{pmatrix}e^{x} & e^{-3x}\\ e^{x} & -3e^{-3x}\end{pmatrix}^{-1}\begin{pmatrix}0\\ 5\sin(3x)\end{pmatrix}=\frac{e^{2x}}{4}\begin{pmatrix}3e^{-3x} & e^{-3x}\\ e^{x} & -e^{x}\end{pmatrix}\begin{pmatrix}0\\ 5\sin(3x)\end{pmatrix}$$
$$=\frac{5}{4}\begin{pmatrix}e^{-x}\sin(3x)\\ -e^{3x}\sin(3x)\end{pmatrix}.$$

So

(1) $$C_1(x)=\frac{5}{4}\int e^{-x}\sin(3x)\,dx \qquad \text{and} \qquad C_2(x)=-\frac{5}{4}\int e^{3x}\sin(3x)\,dx$$

(we don't need integration constants here).

For $a\neq 0$, we calculate $\displaystyle\int \sin(3x)e^{ax}\,dx$ by integrating twice by parts:

$$\int \sin(3x)e^{ax}\,dx=\frac{1}{a}\,e^{ax}\sin(3x)-\frac{1}{a}\int 3\ \cos(3x)e^{ax}\,dx$$
$$=\frac{1}{a}\,e^{ax}\sin(3x)-\frac{3}{a^2}\,e^{ax}\cos(3x)-\frac{3}{a^2}\int 3\sin(3x)e^{ax}\,dx\,,$$

whence, by isolating the integral,

$$\left(1+\frac{9}{a^2}\right)\int \sin(3x)e^{ax}\,dx=\frac{1}{a}\,e^{ax}\ sin(3x)-\frac{3}{a^2}\,e^{ax}\cos(3x)\,,$$

and finally

(2) $$\int \sin(3x)e^{ax}\,dx=\frac{e^{ax}}{a^2+9}\Big(a\sin(3x)-3\cos(3x)\Big)\,.$$

By combining (1) and (2), we find

$$C_1(x)=-\frac{1}{8}\,e^{-x}\big(\sin(3x)+3\cos(3x)\big)\qquad (a=-1)$$
$$C_2(x)=-\frac{5}{24}\,e^{3x}\big(\sin(3x)-\cos(3x)\big)\qquad (a=3)$$

and so the particular solution is

$$y_{\text{part}}(x) = -\frac{1}{3}\sin(3x) - \frac{1}{6}\cos(3x)\,.$$

iii) The general solution of the equation with second term is therefore

$$y(x) = y_{\text{hom}}(x) + y_{\text{part}}(x) = C_1 e^x + C_2 e^{-3x} - \frac{1}{3}\sin(3x) - \frac{1}{6}\cos(3x)\,, \qquad x, C_1, C_2 \in \mathbb{R}\,.$$

To satisfy the initial conditions we must have

$$y(0) = C_1 + C_2 - \frac{1}{6} = 1$$
$$y'(0) = C_1 - 3C_2 - 1 = -\frac{1}{2}$$

The solutions to this system are $C_1 = 1$ and $C_2 = \frac{1}{6}$, so that

$$y(x) = e^x + \frac{1}{6}e^{-3x} - \frac{1}{3}\sin(3x) - \frac{1}{6}\cos(3x)\,, \qquad x \in \mathbb{R}\,.$$

4.

i) By differentiating $y_{\text{part}}(x) = C_1(x)\cos(x) + C_2(x)\sin(x)$, we obtain

$$y'_{\text{part}}(x) = C_1'(x)\cos(x) + C_2'(x)\sin(x) - C_1(x)\sin(x) + C_2(x)\cos(x).$$

By asking that

$$C_1'(x)\cos(x) + C_2'(x)\sin(x) = 0,$$

we then obtain $y'_{\text{part}}(x) = -C_1(x)\sin(x) + C_2(x)\cos(x)$ and

$$\tan(x) = y''_{\text{part}}(x) + y_{\text{part}}(x) = -C_1'(x)\sin(x) + C_2'(x)\cos(x).$$

It is therefore sufficient to solve the system of the statement.

ii) Apply the formula

$$y_{\text{part}}(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W[y_1,y_2](x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W[y_1,y_2](x)}dx,$$

where y_1 and y_2 are two linearly independent solutions of the homogeneous equation, f is the right hand side, the Wronskian $W[y_1, y_2]$ is defined by $W[y_1,y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ and the integration constants can be set freely. Here $f(x) = \tan(x)$ and we can choose $y_1(x) =$

$\cos(x)$, $y_2(x) = \sin(x)$. We find $W[y_1, y_2](x) = 1$ and

$$\begin{aligned} y_{\text{part}}(x) &= -y_1(x) \int \frac{y_2(x) f(x)}{W[y_1, y_2](x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W[y_1, y_2](x)} dx \\ &= -\cos(x) \int \sin(x) f(x) dx + \sin(x) \int \cos(x) f(x) dx \\ &= -\cos(x) \int \frac{\sin^2(x)}{\cos(x)} dx + \sin(x) \int \sin(x) dx \\ &= -\cos(x) \int \frac{(1 - \cos^2(x))}{\cos(x)} dx - \sin(x)\cos(x) \\ &= -\cos(x) \int \frac{1}{\cos(x)} dx. \end{aligned}$$

We complete the calculations as above.

Another method: We solve the system by inverting the matrix :

$$\begin{aligned} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} &= \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \tan(x) \end{pmatrix} \\ &= \frac{1}{\cos(x)^2 + \sin(x)^2} \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix} \begin{pmatrix} 0 \\ \tan(x) \end{pmatrix} = \begin{pmatrix} -\sin(x)\tan(x) \\ \cos(x)\tan(x) \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\sin(x)^2}{\cos(x)} \\ \sin(x) \end{pmatrix} \end{aligned}$$

So we easily get that $C_2(x) = \displaystyle\int \sin(x)\, dx = -\cos(x)$ (remember that we don't need the integration constants for a particular solution).

To find $C_1(x)$ observe that

$$C_1{}'(x) = \frac{-\sin(x)^2}{\cos(x)} = \frac{\cos(x)^2 - 1}{\cos(x)} = \cos(x) - \frac{1}{\cos(x)}$$

and so

$$C_1(x) = \sin(x) - \int \frac{1}{\cos(x)}\, dx\,.$$

To calculate this last primitive, we set the change of variable $t = \sin(x)$. As we work with $x \in (-\pi/2, \pi/2) \subset \text{Im}(\arcsin)$, we have $x = \arcsin(t) := \varphi(t)$ and $\varphi'(t) = \dfrac{1}{\sqrt{1-t^2}}$. Moreover

$\cos(x) > 0$ and therefore $\cos(x) = \sqrt{1-t^2}$. By the formula of the change of variable, we have

$$\begin{aligned}
\int \frac{1}{\cos(x)}\,dx &= \int \frac{1}{\cos(\varphi(t))}\,\varphi'(t)\,dt = \int \frac{1}{\sqrt{1-t^2}}\,\varphi'(t)\,dt \\
&= \int \frac{1}{\sqrt{1-t^2}}\,\frac{1}{\sqrt{1-t^2}}\,dt = \int \frac{1}{1-t^2}\,dt = \int \frac{1}{(1+t)(1-t)}\,dt \\
&= \int \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt = \frac{1}{2}\Big(\ln\left(|1+t|\right) - \ln\left(|1-t|\right)\Big) \\
&= \frac{1}{2}\ln\left(\left|\frac{1+t}{1-t}\right|\right) = \frac{1}{2}\ln\left(\left|\frac{1+\sin(x)}{1-\sin(x)}\right|\right) = \frac{1}{2}\ln\left(\left|\frac{\left(1+\sin(x)\right)^2}{1-\sin(x)^2}\right|\right) \\
&= \frac{1}{2}\ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|^2\right) = \ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|\right)
\end{aligned}$$

Thus, $C_1(x) = \sin(x) - \ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|\right)$ and so

$$\begin{aligned}
y_{\text{part}}(x) &= C_1(x)\cos(x) + C_2(x)\sin(x) \\
&= \left[\sin(x) - \ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|\right)\right]\cos(x) - \cos(x)\sin(x) \\
&= -\cos(x)\ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|\right).
\end{aligned}$$

5. First method: Since $y_1(x) = \frac{1}{x}$ is a solution of the homogeneous equation we are looking for a second solution linearly independent of the homogeneous equation by setting

$$y(x) = C(x)\cdot\frac{1}{x}$$

(method of variation of constants). We get, by substituting

$$y' = C'\frac{1}{x} - C\frac{1}{x^2}\,,\; y'' = C''\frac{1}{x} - 2C'\frac{1}{x^2} + 2C\frac{1}{x^3}$$

in the homogeneous equation $x^2y'' + 3xy' + y = 0$, the differential equation

$$\begin{aligned}
&xC'' - 2C' + 2C\frac{1}{x} + 3C' - 3C\frac{1}{x} + C\frac{1}{x} = 0 \\
&\qquad\qquad \Leftrightarrow xC'' + C' = 0 \Leftrightarrow xu' + u = 0,
\end{aligned}$$

where $u := C'$. This is a first order equation for u. By separating the variables we find $u(x) = \frac{1}{x}$ and therefore, since $x > 0$, $C(x) = \ln(x)$. A second solution of the homogeneous problem is therefore $y_2(x) = \frac{1}{x}\ln(x)$ (we don't need integration constants here). The general solution of the homogeneous problem is therefore

$$y_h(x) = C_1\frac{1}{x} + C_2\frac{1}{x}\ln(x), \quad x > 0,\, C_1,\, C_2 \in \mathbb{R}.$$

We use the method of variation of constants to obtain a particular solution of the inhomogeneous differential equation

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = \frac{2+x^2}{x^2},\ x > 0.$$

Reasoning as in exercise 1 above, we get the system

$$\begin{pmatrix} \frac{1}{x} & \frac{1}{x}\ln(x) \\ -\frac{1}{x^2} & -\frac{1}{x^2}\ln(x) + \frac{1}{x^2} \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2+x^2}{x^2} \end{pmatrix}.$$

The determinant (this is the Wronskian) is

$$-\frac{1}{x^3}\ln(x) + \frac{1}{x^3} + \frac{1}{x^3}\ln(x) = \frac{1}{x^3}$$

and so

$$\begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = x^3 \begin{pmatrix} -\frac{1}{x^2}\ln(x) + \frac{1}{x^2} & -\frac{1}{x}\ln(x) \\ \frac{1}{x^2} & \frac{1}{x} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{2+x^2}{x^2} \end{pmatrix}$$

Which gives

$$\begin{aligned} C_1'(x) &= -\ln(x)(2+x^2) \\ C_2'(x) &= 2+x^2 \end{aligned}$$

and so

$$\begin{aligned} C_1(x) &= -\int \ln(x)(2+x^2)\,dx \\ &= -\ln(x)\left(2x + \frac{1}{3}x^3\right) + \int \left(2 + \frac{1}{3}x^2\right)\,dx \\ &= -\ln(x)\left(2x + \frac{1}{3}x^3\right) + \left(2x + \frac{1}{9}x^3\right) \\ C_2(x) &= 2x + \frac{1}{3}x^3 \end{aligned}$$

(here we can set the integration constants) and we get for the particular solution

$$\begin{aligned} y_{\mathrm{p}}(x) &= C_1(x)y_1(x) + C_2(x)y_2(x) \\ &= -\ln(x)\left(2 + \frac{1}{3}x^2\right) + \left(2 + \frac{1}{9}x^2\right) + \ln(x)\left(2 + \frac{1}{3}x^2\right) \\ &= 2 + \frac{1}{9}x^2. \end{aligned}$$

The general solution of the equation is therefore

$$\begin{aligned} y(x) &= y_{\mathrm{p}}(x) + y_h(x) \\ &= 2 + \frac{1}{9}x^2 + C_1\frac{1}{x} + C_2\frac{1}{x}\ln(x)\ , \quad x > 0,\ C_1,\ C_2\ \in \mathbb{R}. \end{aligned}$$

Second method: applications of formulas. The differential equation is written

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = \frac{2}{x^2} + 1,\ x > 0.$$

Since $y_1(x) = \frac{1}{x}$ for $x > 0$ is a particular solution of the homogeneous equation, a second linearly independent solution is obtained by the formula

$$y_2(x) = y_1(x)\int \frac{e^{-P(x)}}{y_1^2(x)}dx,$$

where $P(x) = \int \frac{3}{x}dx = 3\ln(|x|) = \ln(x^3)$ (we can freely set the integration constants here). We find

$$y_2(x) = \frac{1}{x}\int \frac{1/x^3}{1/x^2}dx = \ln(x)/x.$$

A particular solution for the inhomogeneous equation is given by the formula

$$y_{\text{part}}(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W[y_1,y_2](x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W[y_1,y_2](x)}dx,$$

where $f(x) = \frac{2}{x^2} + 1$, the Wronskian $W[y_1, y_2]$ is defined by $W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ and the integration constants can be set freely. We find

$$W[y_1,y_2](x) = \frac{1}{x}\,\frac{1-\ln(x)}{x^2} + \frac{1}{x^2}\,\frac{\ln(x)}{x} = \frac{1}{x^3}$$

and

$$\begin{aligned}
y_{\text{part}}(x) &= -y_1(x)\int \frac{y_2(x)f(x)}{W[y_1,y_2](x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W[y_1,y_2](x)}dx \\
&= -\frac{1}{x}\int x^3\frac{\ln(x)}{x}\left(\frac{2}{x^2}+1\right)dx + \frac{\ln(x)}{x}\int x^3\frac{1}{x}\left(\frac{2}{x^2}+1\right)dx \\
&= -\frac{1}{x}\int 2\ln(x)dx - \frac{1}{x}\int x^2\ln(x)dx + \frac{\ln(x)}{x}\int\left(2+x^2\right)dx \\
&\overset{p.p.}{=} -\frac{1}{x}\left(2x\ln(x) - \int 2x\frac{1}{x}dx\right) - \frac{1}{x}\left(\frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\cdot\frac{1}{x}dx\right) + \frac{\ln(x)}{x}\left(2x + \frac{x^3}{3}\right) \\
&= -2\ln(x) + 2 - \frac{x^2\ln(x)}{3} + \frac{x^2}{9} + 2\ln(x) + \frac{x^2\ ln(x)}{3} \\
&= 2 + \frac{x^2}{9}\,.
\end{aligned}$$

So the general solution is

$$y(x) = 2 + \frac{1}{9}x^2 + C_1\frac{1}{x} + C_2\frac{1}{x}\ln(x)\ ,\quad x > 0,\ C_1,\ C_2\ \in \mathbb{R}.$$

6. First method: look for a second solution u_2 of the differential equation linearly independent of u_1, of the form $u_2(t) = C(t)u_1(t) = C(t)t$. We obtain $u_2'(t) = C'(t)t + C(t)$ and $u_2''(t) = C''(t)t + 2C'(t)$. By substitution, this gives

$$\left(C''t + 2C'\right) + \frac{1}{t}\left(C't + C\right) - \frac{1}{t^2}Ct = 0$$

$$\Leftrightarrow tC'' + 3C' = 0 \ \Leftrightarrow\ C'' + \frac{3}{t}C' = 0,$$

which is a first-order differential equation for the unknown function C'. The integrating factor is the exponential of any fixed primitive of $\frac{3}{t}$, for example $e^{3\ln(|t|)} = t^3$ (because $t > 0$). So

$$C'' + \frac{3}{t}C' = 0 \Leftrightarrow t^3C'' + 3t^2C' = 0 \Leftrightarrow \left(t^3C'\right)' = 0.$$

By setting $t^3C' = 1$, we obtain

$$C'(t) = \frac{1}{t^3}$$

and we can choose $C(t) = -\frac{1}{2t^2}$. So $u_2(t) = -\frac{1}{2t}$.

Second method: apply the formula

$$u_2(t) = u_1(t)\int \frac{e^{-P(t)}}{u_1^2(t)}dt,$$

where P is any fixed primitive of the function $p(t) = 1/t$, for example $P(t) = \ln(|t|) = \ln(t)$ (because $t > 0$). Moreover, one can freely fix the integration constant. From where $u_2(t) = t\int \frac{1/t}{t^2}dt = t\frac{-1}{2t^2} = -\frac{1}{2t}$.