EXERCISE SHEET 14

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
May 26, 2025

Exercise 1. Give the general solutions to the following differential equations.

i) $y'' + y = 0$
ii) $y'' - y = 0$
iii) $y' = \lambda y$, $\lambda \in \mathbb{R}$ is a constant

Exercise 2. Solve the differential equation

$$3y'' - 4y' + my = 0$$

for each of the following values of the parameter m.

i) $m = 1$
ii) $m = 2$
iii) $m = \frac{4}{3}$

Exercise 3. i) Determine the general solution to the following homogeneous differential equation:

$$y'' + 2y' - 3y = 0\,.$$

ii) By the method of variation of constants, find a particular solution to the following differential equation:

$$y'' + 2y' - 3y = 5\sin(3x)\ . \tag{1}$$

iii) Give a solution to (1) for the initial conditions $y(0) = 1$ and $y'(0) = -\frac{1}{2}$.

Exercise 4. Define $y'' + y = \tan(x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

i) Show that to find a solution particular to the form

$$y_{\text{part}}(x) = C_1(x)\cos(x) + C_2(x)\sin(x)\,,$$

it suffices to solve the system

$$\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \begin{pmatrix} {C_1}'(x) \\ {C_2}'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \tan(x) \end{pmatrix}.$$

ii) Find a solution particular to this system (defined in formula (7.32)).

Exercise 5. Find the general solution $y : (0, \infty) \to \mathbb{R}$ of the differential equation

$$x^2 y'' + 3xy' + y = 2 + x^2\ .$$

i) First method: use the fact that $y_1(x) = \frac{1}{x}$ is a solution of the homogeneous equation and apply the technique of variation of constants twice. (hint: suppose that $y_2(t) = C(t)y_1(t)$)

ii) Second method: notice that $y_1(x) = \frac{1}{x}$ is a solution of the homogeneous equation and apply Theorems 7.13 and 7.14.

Exercise 6. Let

$$u''(t) + \frac{1}{t}u'(t) - \frac{1}{t^2}u(t) = 0,\ t > 0,$$

be a differential equation which admits a particular solution $u_1(t) = t$. Find a second linearly independent solution.

i) First method: variation of constants. (hint: suppose that $u_2(t) = C(t)u_1(t)$)

ii) Second method: apply Theorem 7.13.