EXERCISE SHEET 13 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
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1. All the differential equations in this exercise are linear and of first order.

i) Instead of using the integrating factor method, observe that the left hand side of our equation is the equal to $(xy)'$. Thus, we have

$$xy' + y = \sqrt{x} \quad \iff \quad (xy)' = \left(\frac{2}{3}x^{3/2}\right)' \quad \iff \quad xy = \frac{2}{3}x^{3/2} + c,$$

for some $c \in \mathbb{R}$. Hence we have

$$y = \frac{2}{3}\sqrt{x} + \frac{c}{x}.$$

We remark that even if we used the integrating factor method, the equation would remain the same, thus we avoided a few extra steps in this way.

ii) As in i), we have

$$(\sin x)y' + (\cos x)y = \sin(x^2) \quad \iff \quad ((\sin x)y)' = \left(\int_0^x \sin(t^2)\underset{.}{t}\right)' (\sin x)y = \int_0^x \sin(t^2)\underset{.}{t} + c,$$

for some $c \in \mathbb{R}$. Hence we have

$$y = \frac{1}{\sin x}\int_0^x \sin(t^2)\underset{.}{t} + \frac{c}{\sin x}.$$

iii) We rewrite the equation as

$$y' - \frac{4}{x}y = x^4 e^x.$$

The integrating factor is

$$e^{-\int \frac{4}{x}\underset{.}{X}} = e^{-\ln x} = e^{\ln\left(\frac{1}{x^4}\right)} = \frac{1}{x^4}.$$

Multiplying the equation by the integrating factor, it becomes

$$\frac{y'}{x^4} - \frac{4y}{x^5} = e^x \quad \iff \quad \left(\frac{y}{x^4}\right)' = (e^x)' \quad \iff \quad \frac{y}{x^4} = e^x + c,$$

for some $c \in \mathbb{R}$. Thus, we have

$$y = x^4 e^x + cx^4.$$

iv) We rewrite the equation as

$$y' + \frac{y}{x \ln x} = \frac{1}{x \ln^3 x}.$$

The integrating factor is

$$e^{\int \frac{X}{x \ln x}} = e^{\ln \ln x} = \ln x.$$

Multiplying by this, we have

$$(\ln x)y' + \frac{y}{x} = \frac{1}{x \ln^2 x} \iff ((\ln x)y)' = \left(-\frac{1}{\ln x}\right)',$$

which finally gives

$$y = \frac{c}{\ln x} - \frac{1}{\ln^2 x}.$$

2. All the differential equations in this exercise are linear and of first order.

i) we have

$$x^3y' + 3x^2y = \cos x \iff (x^3y)' = (\sin x)' \iff x^3y = \sin x + c,$$

for some $c \in \mathbb{R}$. Using that $y(\pi) = 0$, we find $c = 0$. Hence we have

$$y = \frac{\sin x}{x^3}.$$

ii) We rewrite the equation as

$$y' + \frac{y}{2x} = 3.$$

The integrating factor is

$$e^{\int \frac{X}{2x}} = e^{\frac{1}{2}\ln x} = e^{\ln \sqrt{x}} = \sqrt{x}.$$

Multiplying by this, we have

$$\sqrt{x}y' + \frac{y}{2\sqrt{x}} = 3\sqrt{x} \iff (\sqrt{x}y)' = (2x^{3/2})' \iff \sqrt{x}y = 2x^{3/2} + c,$$

for some $c \in \mathbb{R}$. Using that $y(4) = 20$, we find $c = 24$. Thus, we have

$$y = 2x + \frac{24}{\sqrt{x}}.$$

iii) Using the integrating factor method as before, we find

$$(x^2+1)^{3/2}y = (x^2+1)^{3/2} + c,$$

for some $c \in \mathbb{R}$. Using that $y(0) = 2$, we find $c = 1$ and then we have

$$y = 1 + (x^2+1)^{-3/2}.$$

3.

i) Since we want $x > 0$ and $y(x) > 0$ (see the statement), it is a Bernoulli differential equation:

$$y' - \frac{y}{x} = -\frac{x}{y}\,,$$

the second member being $-xy^\alpha(x)$ with $\alpha = -1$. Multiply the left and right by $(y(x))^{-\alpha} = y(x)$:

$$yy' - \frac{y^2}{x} = -x \ \Leftrightarrow\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(y^2(x)) - \frac{y^2}{x} = -x.$$

Let $z = y^2 > 0$, which gives $z' - \dfrac{2}{x}z = -2x$. This is a first order linear equation for the function z, which we solve here by the method of the integrating factor. The integrating factor is $\exp\left(\int \frac{-2\mathrm{d}x}{x}\right) = \frac{1}{x^2}$, so

$$z' - \frac{2}{x}z = -2x \ \Leftrightarrow\ \frac{1}{x^2}z' - \frac{2}{x^3}z = -\frac{2}{x} \ \Leftrightarrow\ \left(\frac{1}{x^2}z\right)' = -\frac{2}{x}$$

$$\Leftrightarrow\ \frac{1}{x^2}z = -2\ln(|x|) + A = \ln(e^A x^{-2})$$

Since we want $x > 0$ and $y(x) > 0$, we obtain

$$z = x^2 \ln\frac{e^A}{x^2} = y^2(x),\ \ A \in \mathbb{R},$$

and

$$y(x) = x\sqrt{\ln\frac{C}{x^2}}\,,\ \ 0 < x < \sqrt{C}$$

where $C > 0$ is a constant.

ii) Solving as in i), we find

$$y = \frac{1}{x(c + \ln x)},$$

for some constant $c \in \mathbb{R}$.

iii) We divide both sides of the equation by y^3 to get

$$\frac{y'}{y^3} + \frac{2}{xy^2} = \frac{1}{x},$$

which means that

$$(-1/2)dy^{-2} + \frac{2}{xy^2} = \frac{1}{x}.$$

Now we set $z = y^{-2}$ to obtain $dz = \frac{4z}{x} - \frac{2}{x} \Leftrightarrow \frac{dz}{4z-2} = \frac{dx}{x}$. By solving this equation, we have

$$\frac{1}{4}\log(z - \frac{1}{2}) = \log x + C \ \Leftrightarrow\ z - \frac{1}{2} = Cx^4.$$

Therefore, $y = \pm(\frac{1}{2} + Cx^4)^{-1/2}$.

4.

a) Let us first notice that the differential equation is of the form $y' = h(y)$ with $h(y) = 5(y^4)^{1/5}$, where the function $s \to s^{1/5}$ for $s \in \mathbb{R}$ is the reciprocal function of the function $t \to t^5$ for $t \in \mathbb{R}$.

The function $y : \mathbb{R} \to \mathbb{R}$ defined by $y(x) = 0$ for $x \leq 0$ and $y(x) = x^5$ for $x > 0$ satisfies the differential equation at all $x \neq 0$. Moreover y is continuous at 0 with $y(0) = 0$. It is differentiable at 0:

$$\lim_{x\to 0-} \frac{y(x) - y(0)}{x - 0} = \lim_{x\to 0-} \frac{0 - 0}{x} = \lim_{x\to 0-} 0 = 0,$$

$$\lim_{x\to 0+} \frac{y(x) - y(0)}{x - 0} = \lim_{x\to 0+} \frac{x^5 - 0}{x} = 0 = \lim_{x\to 0-} \frac{y(x) - y(0)}{x - 0}$$

and therefore $y'(0) = 0$. The differential equation is also satisfied at $x = 0$: $y'(0) = 0 = 5(0^4)^{1/5} = 5(y(0)^4)^{1/5}$. Finally y' is continuous because $y'(x) = 0$ for $x \leq 0$ and $y'(x) = 5x^4$ for $x > 0$.

b) The differential equation $y' = 5\left(y^4\right)^{1/5}$ has separated variables. Under the assumption $y(x) \neq 0$, we get by integration

$$\frac{dy}{5\left(y^4\right)^{1/5}} = dx \qquad \Leftrightarrow \qquad y^{1/5} = x - C, \quad C \in \mathbb{R} \qquad \Leftrightarrow \qquad y = (x - C)^5, \quad C \in \mathbb{R}.$$

Moreover, we have the trivial solution $y = 0$.

With this, we have not yet found all the solutions of the given equation. What is missing are the piecewise combinations of the obtained solutions; we need to combine them in such a way that the result is a continuously differentiable function. The three elements that can be combined are therefore the positive and negative parts of the function $(x - C)^5$ for distinct values of C, and the trivial function $y = 0$. The set of maximal solutions is thus given by

$$y(x) = \begin{cases} (x - C_1)^5 & x < C_1 \\ 0 & C_1 \leq x \leq C_2 \\ (x - C_2)^5 & x > C_2 \end{cases} \qquad \text{where} \quad C_1, C_2 \in \mathbb{R}, \quad C_1 \leq C_2 \tag{1}$$

$$y(x) = \begin{cases} (x - C)^5 & x < C \\ 0 & x \geq C \end{cases} \qquad \text{where} \quad C \in \mathbb{R}$$

$$y(x) = \begin{cases} 0 & x \leq C \\ (x - C)^5 & x > C \end{cases} \qquad \text{where} \quad C \in \mathbb{R}$$

$$y(x) = 0, \quad x \in \mathbb{R}$$

Let us denote here and below by $\widetilde{y}$ the solution obtained by integration: $\widetilde{y}(x) = (x - C)^5$. It corresponds to (1) with $C_1 = C_2 = C$.

c) For $\widetilde{y}(-3) = (-3 - C)^5 = -1$, $C = -2$ and for $\widetilde{y}(2) = (2 - C)^5 = 1$, we need $C = 1$. The solution sought is therefore (1) with $C_1 = -2$ and $C_2 = 1$.

5.

i) $y(x) = C_1 x + C_2$ (integration)

ii) $y(x) = \frac{1}{2}x^2 + C_1 x + C_2$ (integration)