# EXERCISE SHEET 11 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
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**1**.

i) We use spherical coordinates:

$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi \end{cases},$$

and then $x^2+y^2+z^2 \le 4$ implies $0 \le r \le 2$, and $y \ge 0$ implies $0 \le \phi, \theta \le \pi$. Recall that the Jacobian determinant is $r^2\sin\phi$ and then we have

$$\begin{aligned}\iiint\limits_E (x^2+y^2)\mathrm{xyz} &= \int_0^\pi\int_0^\pi\int_0^2 \left((r\sin\phi\cos\theta)^2 + (r\sin\phi\sin\theta)^2\right) r^2\sin\phi\mathrm{r}\theta\phi \\ &= \int_0^\pi\int_0^\pi\int_0^2 r^4\sin^3\phi\mathrm{r}\theta\phi \\ &= \pi\left(\int_0^2 r^4\mathrm{r}\right)\left(\int_0^\pi \sin^3\phi\phi\right) = \frac{128\pi}{15},\end{aligned}$$

since the integral on $r$ is $\frac{32}{5}$, while the other one is

$$\int_0^\pi \sin^3\phi\phi = \int_0^\pi \sin\phi(1-\cos^2\phi)\phi = \left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Bigg|_0^\pi = \frac{4}{3}.$$

ii) $E$ is the region between the sphere $x^2+y^2+z^2 \le 1$ and the cone $z \ge \sqrt{x^2+y^2}$. We use spherical coordinates as in i). Then the first condition gives $0 \le r \le 1$, while the second one gives

$$r\cos\phi \ge r\sin\phi \iff \tan\phi \le 1 \iff 0 \le \phi \le \frac{\pi}{4}.$$

Therefore, we have

$$\begin{aligned}\iiint\limits_E 3z\mathrm{xyz} &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 3r\cos\phi\cdot r^2\sin\phi\mathrm{r}\theta\phi \\ &= 6\pi\left(\int_0^1 r^3\mathrm{r}\right)\left(\int_0^{\frac{\pi}{4}}\cos\phi\sin\phi\phi\right) = 6\pi\cdot\frac{r^4}{4}\Bigg|_0^1\cdot\frac{\sin^2\phi}{2}\Bigg|_0^{\frac{\pi}{4}} = \frac{3\pi}{8}.\end{aligned}$$

iii) The region $E$ is given by

$$\begin{cases} 4 \leq x^2 + z^2 \leq 9 \\ 1 \leq y \leq 5 \\ z \leq 0 \end{cases}.$$

We use cylindrical coordinates. We set

$$\begin{cases} x = r\cos\theta \\ y = y \\ z = r\sin\theta \end{cases}.$$

Then the new region in cylindrical coordinates is given by

$$\begin{cases} 2 \leq r \leq 3 \\ 1 \leq y \leq 5 \\ \pi \leq \theta \leq 2\pi \end{cases}.$$

The Jacobian determinant is $r$, hence we have

$$\iiint\limits_E e^{-x^2-z^2}\,\text{\d{x}\d{y}\d{z}} = \int_\pi^{2\pi}\int_1^5\int_2^3 re^{-r^2}\,\text{\d{r}\d{y}}\underset{\cdot}{\theta} = 4\pi\int_2^3 re^{-r^2}\,\text{\d{r}} = 4\pi\cdot\frac{-e^{-r^2}}{2}\Bigg|_2^3$$
$$= 2\pi(e^{-4} - e^{-9}).$$

**2**. We have that the volume of $E$ is

$$V = \int_0^2\int_0^1\int_0^{4-xy}\text{\d{z}\d{y}\d{x}} = \int_0^2\int_0^1(4-xy)\,\text{\d{y}\d{x}} = \int_0^2\left(4y - \frac{xy^2}{2}\right)\Bigg|_0^1\text{\d{x}}$$
$$= \int_0^2\left(4 - \frac{x}{2}\right)\text{\d{x}} = \left(4x - \frac{x^2}{4}\right)\Bigg|_0^2 = 7.$$

**3**. Observing the figure, we deduce that the top of the domain (gray part in Fig. 1) belongs to the plane with the equation $y + 2z = 2$ .

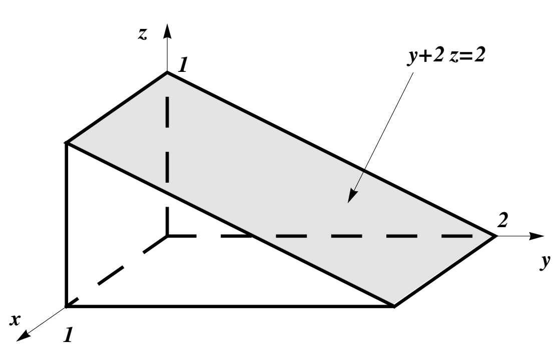


FIGURE 1

Thus the bounds of the domain $S$ are

$$0 \leq x \leq 1\,, \qquad 0 \leq y \leq 2 \qquad \text{and} \qquad 0 \leq z \leq \frac{2-y}{2}\,.$$

The total mass is therefore

$$\begin{aligned}
M &= \iiint\limits_S \delta(x,y,z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^2 \left(\int_0^{(2-y)/2} 4x^2\,dz\right) dy\right) dx \\
&= 4\int_0^1 x^2 \left(\int_0^2 \left(\int_0^{(2-y)/2} dz\right) dy\right) dx = 4\left(\int_0^2 \left(\int_0^{(2-y)/2} dz\right) dy\right)\left(\int_0^1 x^2\,dx\right) \\
&= 2\left(\int_0^2 (2-y)\,dy\right)\left(\int_0^1 x^2\,dx\right) = 2\cdot\left[2y - \frac{1}{2}y^2\right]_0^2 \cdot \left[\frac{1}{3}x^3\right]_0^1 = 2\cdot 2\cdot\frac{1}{3} = \frac{4}{3}\,.
\end{aligned}$$

For the center of mass, we have

$$\begin{aligned}
M_{yz} &= \iiint\limits_S x\cdot\delta(x,y,z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^2 \left(\int_0^{(2-y)/2} 4x^3\,dz\right) dy\right) dx \\
&= 4\left(\int_0^2 \left(\int_0^{(2-y)/2} dz\right) dy\right)\left(\int_0^1 x^3\,dx\right) = 4\cdot\left[\frac{1}{4}x^4\right]_0^1 = 4\cdot\frac{1}{4} = 1\,, \\
M_{xz} &= \iiint\limits_S y\cdot\delta(x,y,z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^2 \left(\int_0^{(2-y)/2} 4x^2 y\,dz\right) dy\right) dx \\
&= 4\left(\int_0^2 \left(\int_0^{(2-y)/2} y\,dz\right) dy\right)\left(\int_0^1 x^2\,dx\right) = 2\left(\int_0^2 y(2-y)\,dy\right)\cdot\frac{1}{3} \\
&= 2\cdot\left[y^2 - \frac{1}{3}y^3\right]_0^2\cdot\frac{1}{3} = 2\cdot\frac{4}{3}\cdot\frac{1}{3} = \frac{8}{9}\,, \\
M_{x,y} &= \iiint\limits_S z\cdot\delta(x,y,z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^2 \left(\int_0^{(2-y)/2} 4x^2 z\,dz\right) dy\right) dx \\
&= 4\left(\int_0^2 \left(\int_0^{(2-y)/2} z\,dz\right) dy\right)\left(\int_0^1 x^2\,dx\right) = 4\left(\int_0^2 \left[\frac{1}{2}z^2\right]_0^{(2-y)/2} dy\right)\cdot\frac{1}{3} \\
&= \frac{1}{2}\left(\int_0^2 (2-y)^2\,dy\right)\cdot\frac{1}{3} = \frac{1}{2}\left[-\frac{1}{3}(2-y)^3\right]_0^2\cdot\frac{1}{3} = \frac{1}{2}\cdot\frac{8}{3}\cdot\frac{1}{3} = \frac{4}{9}\,.
\end{aligned}$$

Thus the center of mass is $(\overline{x},\overline{y},\overline{z}) = \left(\frac{3}{4},\,\frac{2}{3},\,\frac{1}{3}\right)$.

**4**. The domain $D$ is given by the inequalities

$$0 \leq z \leq 1\,, \qquad 0 \leq y \leq z \qquad \text{and} \qquad 0 \leq x \leq \sqrt{y}\,,$$

and then the total mass is

$$
\begin{aligned}
M &= \int_0^1 \left( \int_0^z \left( \int_0^{\sqrt{y}} z^{7/2}\, e^{-y^{3/2} z^{3/2}}\, dx \right) dy \right) dz \\
&= \int_0^1 \left( \int_0^z \left[ z^{7/2}\, e^{-y^{3/2} z^{3/2}} x \right]_{x=0}^{x=\sqrt{y}} dy \right) dz = \int_0^1 \left( \int_0^z z^{7/2}\, e^{-y^{3/2} z^{3/2}} \sqrt{y}\, dy \right) dz \\
&= \int_0^1 \left( \int_0^z z^{7/2} \left( -\frac{2}{3} \frac{1}{z^{3/2}} \right) \cdot \underbrace{\left( -\frac{3}{2} z^{3/2} y^{1/2} \right) e^{-y^{3/2} z^{3/2}}}_{=\varphi'(y)\exp(\varphi(y))} dy \right) dz \\
&= \int_0^1 \left[ -\frac{2}{3} \frac{z^{7/2}}{z^{3/2}} e^{-y^{3/2} z^{3/2}} \right]_{y=0}^{y=z} dz = -\frac{2}{3} \int_0^1 \left[ z^2\, e^{-y^{3/2} z^{3/2}} \right]_{y=0}^{y=z} dz \\
&= -\frac{2}{3} \int_0^1 \left( z^2\, e^{-z^3} - z^2 \right) dz = -\frac{2}{3} \int_0^1 \left( z^2\, e^{-z^3} - z^2 \right) dz \\
&= -\frac{2}{3} \left[ -\frac{1}{3} e^{-z^3} - \frac{1}{3} z^3 \right]_0^1 = \frac{2}{9e}.
\end{aligned}
$$

**5.**

i) We have

$$
y'(x) = \pm \frac{1}{2\sqrt{e^{\left(\frac{C}{x-1}\right)} - 1}} \frac{-C}{(x-1)^2}\, e^{\left(\frac{C}{x-1}\right)} = \frac{1}{2y(x)} \frac{-C}{(x-1)^2}\, e^{\left(\frac{C}{x-1}\right)},
$$

and

$$
y(x)^2 + 1 = e^{\left(\frac{C}{x-1}\right)}.
$$

So

$$
\begin{aligned}
y'(x) \frac{2y(x)(x-1)}{y(x)^2+1} + \ln\left(y(x)^2 + 1\right) &= \frac{1}{2y(x)} \frac{-C}{(x-1)^2}\, e^{\left(\frac{C}{x-1}\right)} \frac{2y(x)(x-1)}{e^{\left(\frac{C}{x-1}\right)}} + \ln\left(e^{\left(\frac{C}{x-1}\right)}\right) \\
&= \frac{-C}{x-1} + \ln\left(e^{\left(\frac{C}{x-1}\right)}\right) = \frac{-C}{x-1} + \frac{C}{x-1} = 0,
\end{aligned}
$$

that is, the function $y(x)$ satisfies the given differential equation.

With $y(2) = -3$, we are in the case where $x > 1$ and $y(x)$ is given by the negative root. We have

$$
y(2) = -\sqrt{e^C - 1} = -3 \qquad \iff \qquad e^C - 1 = 9 \qquad \iff \qquad C = \ln(10) > 0.
$$

When $y\left(-\frac{3}{2}\right) = 2$, we are in the case where $x < -1$ and $y(x)$ is the positive root. We have

$$
y\left(-\tfrac{3}{2}\right) = \sqrt{e^{\left(-\frac{2C}{5}\right)} - 1} = 2 \qquad \iff \qquad e^{\left(-\frac{2C}{5}\right)} = 5 \qquad \iff \qquad C = -\tfrac{5}{2}\ln(5) < 0.
$$

ii) We have

$$y'(x) = 1 - \frac{2(1+xe^{-x}) - 2x(e^{-x} - xe^{-x})}{(1+xe^{-x})^2} = \underbrace{1 - \frac{2}{1+xe^{-x}}}_{=\dfrac{y(x)}{x}} + \frac{2xe^{-x}(1-x)}{(1+xe^{-x})^2}\ ,$$

and

$$y(x)^2 - x^2 = \left(x - \frac{2x}{1+xe^{-x}}\right)^2 - x^2 = -\frac{4x^2}{1+xe^{-x}} + \frac{4x^2}{\left(1+xe^{-x}\right)^2}$$
$$= \frac{-4x^2\left(1+xe^{-x}\right) + 4x^2}{(1+xe^{-x})^2} = \frac{-4x^3e^{-x}}{(1+xe^{-x})^2}\ .$$

So

$$2x^2\, y'(x) = 2x\, y(x) - \frac{4x^3e^{-x}(x-1)}{(1+xe^{-x})^2}\ ,$$

and so

$$2x^2\, y'(x) - (x-1)(y(x)^2 - x^2) - 2x\, y(x) = -\frac{4x^3e^{-x}(x-1)}{(1+xe^{-x})^2} - (x-1)\frac{-4x^3e^{-x}}{(1+xe^{-x})^2} = 0\,.$$

The function $y(x)$ therefore satisfies the differential equation.

**6**.

i) By integrating we have

$$y(x) = \int \ln x \underset{\cdot}{\mathrm{x}} + \int \tan x \underset{\cdot}{\mathrm{x}} + C_1,$$

for some $C_1 \in \mathbb{R}$. For the first integral we use integration by parts and we have

$$\int \ln x \underset{\cdot}{\mathrm{x}} = x\ln x - \int x(\ln x)' \underset{\cdot}{\mathrm{x}} = x\ln x - \int \underset{\cdot}{\mathrm{x}} = x\ln x - x + C_2,$$

for some $C_2 \in \mathbb{R}$, while for the second one we write

$$\tan x = \frac{\sin x}{\cos x} = -\frac{(\cos x)'}{\cos x} = -(\ln\cos x) \Longrightarrow \int \tan x \underset{\cdot}{\mathrm{x}} = -(\ln\cos x) + C_3,$$

for some $C_3 \in \mathbb{R}$ Hence

$$y = x\ln x - x - \ln\cos x + C,$$

for some $C \in \mathbb{R}$.

ii) By integrating we have

$$y(x) = \int \sin x \cdot e^{\cos x} \underset{\cdot}{\mathrm{x}} = -\int (\cos x)' e^{\cos x} \underset{\cdot}{\mathrm{x}} = -\int (e^{\cos x})' \underset{\cdot}{\mathrm{x}} = -e^{\cos x} + C,$$

for some $C \in \mathbb{R}$.

iii) By integrating we have

$$y(x) = \int 2x\sqrt{x^2+16}\underset{\cdot}{x} = \int (x^2+16)'\sqrt{x^2+16}\underset{\cdot}{x} = \frac{2}{3}\int \left((x^2+16)^{3/2}\right)' \underset{\cdot}{x}$$
$$= \frac{2}{3}(x^2+16)^{3/2} + C,$$

for some $C \in \mathbb{R}$.

iv) By integrating we have

$$y(x) = \pi + \int_{1/2}^{x} \frac{dt}{\sqrt{t-t^2}} = \pi + \int_{1/2}^{x} \frac{dt}{\sqrt{t}\sqrt{1-t}}.$$

Let's consider the change of variable $t = \varphi(u) = u^2$, $\varphi'(u) = 2u$ with $u > 0$. Since the bounds of $t$ are $1/2$ and $x$, and since $\varphi(\sqrt{2}/2) = 1/2$ and $\varphi(\sqrt{x}) = x$, the bounds of $u$ are $\sqrt{2}/2$ and $\sqrt{x}$. So

$$y(x) = \pi + \int_{\sqrt{2}/2}^{\sqrt{x}} \frac{2u\,du}{u\sqrt{1-u^2}} = \pi + \int_{\sqrt{2}/2}^{\sqrt{x}} \frac{2\,du}{\sqrt{1-u^2}} = \pi + 2\arcsin(\sqrt{x}) - 2\arcsin(\sqrt{2}/2)$$
$$= (\pi/2) + 2\arcsin(\sqrt{x})$$

v) For $x < 2$, $y'(x) = 2 - x \implies y(x) = 2x - \frac{x^2}{2} + C$. With $y(1) = 3$, we get $C = 3/2$ and $y(x) = 2x - \frac{x^2}{2} + \frac{3}{2}$. As we are looking for $y$ continuous in $x = 2$, we get $y(2) = \frac{7}{2}$.
For $x > 2$, $y'(x) = x - 2 \implies y(x) = -2x + \frac{x^2}{2} + d$. With $y(2) = \frac{7}{2}$, we get $d = \frac{11}{2}$. From where

$$y(x) = \begin{cases} 2x - \frac{x^2}{2} + \frac{3}{2} & x \leq 2 \\ -2x + \frac{x^2}{2} + \frac{11}{2} & x \geq 2. \end{cases}$$

**7**.

i) We have

$$y' - y = 0 \iff y'e^{-x} - ye^{-x} = 0 \iff (ye^{-x})' = 0 \iff ye^{-x} = C \quad \text{for some } C \in \mathbb{R}$$
$$\iff y = Ce^{x}.$$

ii) We have

$$y' + 2y = 0 \iff y'e^{2x} + 2ye^{2x} = 0 \iff (ye^{2x})' = 0 \iff ye^{2x} = C \quad \text{for some } C \in \mathbb{R}$$
$$\iff y = Ce^{-2x}.$$

iii) We have

$$y' + 2xy = x \iff e^{x^2}y' + 2xe^{x^2}y = xe^{x^2} \iff (e^{x^2}y)' = \left(\frac{e^{x^2}}{2}\right)'$$
$$\iff e^{x^2}y = \frac{e^{x^2}}{2} + C \quad \text{for some } C \in \mathbb{R} \iff y = \frac{1}{2} + Ce^{-x^2}.$$

iv) We have

$$\begin{aligned}
xy' + \frac{y}{x} = e^{1/x} \quad &\iff \quad y' + \frac{y}{x^2} = \frac{e^{1/x}}{x} \quad \iff \quad e^{-1/x}y' + \frac{e^{-1/x}}{x^2}y = \frac{1}{x} \\
&\iff \quad (e^{-1/x}y)' = (\ln x)' \quad \iff \quad e^{-1/x}y = \ln x + C \quad \text{for some } C \in \mathbb{R} \\
&\iff \quad y = e^{1/x}\ln x + Ce^{1/x}.
\end{aligned}$$