EXERCISE SHEET 11

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
May 5, 2025

Exercise 1. Calculate the following triple integrals using either spherical or cylindrical coordinates:

i) $\iiint\limits_E (x^2+y^2)\underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}}$, where $E=\{(x,y,z)\in\mathbb{R}^3\colon x^2+y^2+z^2\le 4,\ y\ge 0\}$.

ii) $\iiint\limits_E 3z\underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}}$, where E is the region inside both $x^2+y^2+z^2=1$ and $z=\sqrt{x^2+y^2}$.

iii) $\iiint\limits_E e^{-x^2-z^2}\underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}}$, where E is the region between the two cylinders $x^2+z^2=4,\ x^2+z^2=9$ with $1\le y\le 5$ and $z\le 0$.

Exercise 2. Suppose that E is the region below $z=4-xy$ and above the region in the xy-plane defined by $0\le x\le 2$ and $0\le y\le 1$. Calculate the volume of E.

Exercise 3. Determine the center of mass $(\overline{x},\overline{y},\overline{z})$ of the domain S represented in Fig. 1 if the mass density is given by $\delta\colon D\to\mathbb{R}$ is $\delta(x,y,z)=4x^2$.

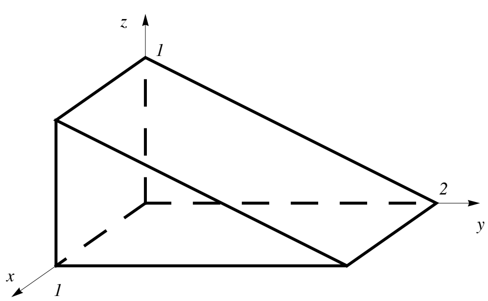

FIGURE 1

Exercise 4. Let $D=\{(x,y,z)\in\mathbb{R}^3 \mid 0\le x\le 1,\ x^2\le y\le 1,\ y\le z\le 1\}$. Calculate the total mass of D if the mass density $\delta\colon D\to\mathbb{R}$ is given by

$$\delta(x,y,z)=z^{7/2}\ e^{-y^{3/2}z^{3/2}}.$$

Exercise 5. Suppose that $y=y(x)$ is given in the following form. Verify that the function $y(x)$ satisfies the corresponding differential equation in each case:

i) Let $y(x) = \pm\sqrt{e^{\left(\frac{C}{x-1}\right)} - 1}$. Show that this function satisfies the differential equation $y' \dfrac{2y(x-1)}{y^2+1} + \ln(y^2+1) = 0$, for respectively, $x > 1$ and $C > 0$; and $x < 1$ and $C < 0$. Then determine the values of C so that $y(2) = -3$ and $y\left(-\frac{3}{2}\right) = 2$, respectively.

ii) (Riccati equation) Let $y(x) = x - \dfrac{2x}{1 + xe^{-x}}$. Show that this function satisfies the differential equation $2x^2y' = (x-1)(y^2 - x^2) + 2xy$, for $x > 0$.

Exercise 6. Solve the following differential equations:

i) $y' = \ln x + \tan x$.
ii) $y' = \sin x \cdot e^{\cos x}$.
iii) $y' = 2x\sqrt{x^2+16}$.
iv) $y' = \dfrac{1}{\sqrt{x - x^2}}$, $y(1/2) = \pi$ (hint: arcsin is involved in the solution).
v) $y' = |x-2|$, $y(1) = 3$ (hint: find a solution which is defined for all $\mathbb{R}$).

Exercise 7. Solve the following differential equations:

i) $y' = y$.
ii) $y' + 2y = 0$.
iii) $y' + 2xy = x$.
iv) $xy' + \frac{y}{x} = e^{1/x}$, $x > 0$.

(Hint: Consider using the method described in Exercise 7.7.)