EXERCISE SHEET 10 SOLUTIONS

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
April 28, 2025

1.

i) The equations of the line between:
   - $(2,0)$ and $(5,3)$ is $y = x - 2$
   - $(5,3)$ and $(6,7)$ is $y = 4x - 17$
   - $(6,7)$ and $(3,4)$ is $y = x + 1$
   - $(3,4)$ and $(2,0)$ is $y = 4x - 8$

   so these four lines define the parallelogram D. By applying the transformation $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(4v - u)$ to these lines, we get
   - $y = x - 2$: $v = -2$
   - $y = 4x - 17$: $u = -17$
   - $y = x + 1$: $v = 1$
   - $y = 4x - 8$: $u = -8$

   Now the Jacobian determinant of this trnasformation is

$$\left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{pmatrix} \right| = -\frac{1}{3}.$$

   Therefore, we have

$$\begin{aligned} \iint_D (6x - 3y)\underset{\cdot}{x}\underset{\cdot}{y} &= \int_{-17}^{-8} \int_{-2}^{1} \left(6 \cdot \frac{1}{3}(v - u) - 3 \cdot \frac{1}{3}(4v - u) \right) \left| -\frac{1}{3} \right| \underset{\cdot}{v}\underset{\cdot}{u} \\ &= -\frac{1}{3} \int_{-17}^{-8} \int_{-2}^{1} (2v + u)\underset{\cdot}{v}\underset{\cdot}{u} = -\frac{1}{3} \int_{-17}^{-8} (v^2 + uv)\Big|_{-2}^{1} \underset{\cdot}{u} \\ &= \int_{-17}^{-8} (1 - u)\underset{\cdot}{u} = \left(u - \frac{u^2}{2} \right) \Big|_{-17}^{-8} = \frac{243}{2}. \end{aligned}$$

ii) As in i), we first find the curves defining the region D and the we apply the transformation $x = \frac{v}{6u}$, $y = 2u$ to obtain:
   - $y = 2$: $u = 1$
   - $xy = 3$: $v = 9$
   - $y = 6$: $u = 3$
   - $xy = 1$: $v = 3$

The Jacobian determinant is

$$\left|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\begin{pmatrix} -\frac{v}{6u^2} & \frac{1}{6u} \\ 2 & 0 \end{pmatrix}\right| = -\frac{1}{3u}.$$

Therefore, we have

$$\iint\limits_D xy^3 \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}} = \int_1^3 \int_3^9 \frac{v}{6u}(2u)^3 \left|-\frac{1}{3u}\right| \underset{\cdot}{\mathrm{v}}\underset{\cdot}{\mathrm{u}} = \frac{4}{9}\int_1^3 u \int_3^9 v \underset{\cdot}{\mathrm{v}}\underset{\cdot}{\mathrm{u}} = \frac{4}{9} \cdot \frac{u^2}{2}\Big|_1^3 \cdot \frac{v^2}{2}\Big|_3^9 = 64.$$

2.

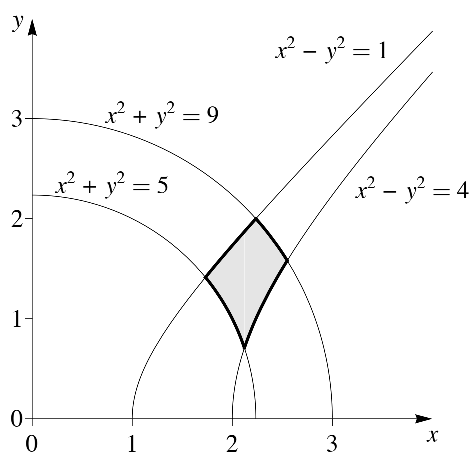

FIGURE 1

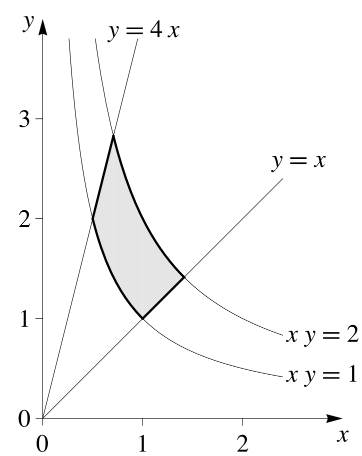

FIGURE 2

i) The domain D is represented in Fig. 1 (we choose it to be closed). For the change of variables, we define the mapping $H: D \to \widetilde{D}$ such that $(u, v) = H(x, y)$ with

$$\begin{cases} u = x^2 + y^2 = H_1(x, y) \\ v = x^2 - y^2 = H_2(x, y) \end{cases}$$

and $x, y \geq 0$. It follows from the definition of D that $\widetilde{D} = [5, 9] \times [1, 4]$. Furthermore, $2x^2 = u + v$, $2y^2 = u - v$ (hence expressions for x and y) and $4x^2y^2 = u^2 - v^2 \geq 25 - 16 > 0$. The Jacobian matrix H is

$$J_H(x, y) = \begin{pmatrix} \partial_x H_1(x, y) & \partial_y H_1(x, y) \\ \partial_x H_2(x, y) & \partial_y H_2(x, y) \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

and its determinant is $\det\big(J_H(x, y)\big) = -8xy < 0$.

Let $G = H^{-1}: \widetilde{D} \to D$ be the inverse transformation such that $(x, y) = G(u, v)$. For the calculation of the integral, we need the determinant of the Jacobian of G which is

$$\det\big(J_G(u, v)\big) = \left[\frac{1}{\det\big(J_H(x, y)\big)}\right]_{(x,y)=G(u,v)} = \left[-\frac{1}{8xy}\right]_{(x,y)=G(u,v)} < 0.$$

By applying the change of variables formula to the open $\mathring{D}$ and $\mathring{\widetilde{D}}$, we find

$$\begin{aligned}
\iint_D x^3y^3\,dx\,dy &= \iint_{\mathring{D}} x^3y^3\,dx\,dy = \iint_{\mathring{\widetilde{D}}} \Big[x^3y^3\Big]_{(x,y)=G(u,v)} \cdot \left|\det\left(J_G(u,v)\right)\right|\,du\,dv \\
&= \iint_{\widetilde{D}} \left[x^3y^3\cdot\frac{1}{8xy}\right]_{(x,y)=G(u,v)} du\,dv = \frac{1}{8}\iint_{\widetilde{D}} \Big[x^2y^2\Big]_{(x,y)=G(u,v)} du\,dv \\
&= \frac{1}{32}\int_1^4 \left(\int_5^9 (u^2-v^2)\,du\right)dv = \frac{1}{32}\int_1^4 \left[\frac{1}{3}u^3 - uv^2\right]_{u=5}^{u=9} dv \\
&= \frac{1}{32}\int_1^4 \left(\frac{9^3-5^3}{3} - 4v^2\right)dv = \frac{1}{24}\int_1^4 (151-3v^2)\,dv \\
&= \frac{1}{24}\Big[151v - v^3\Big]_1^4 = \frac{390}{24} = \frac{65}{4}\,.
\end{aligned}$$

ii) The domain D (closed) is in the first quadrant (because $x, y \geq 0$) and is bounded by the lines $y = x$ and $y = 4x$ and the curves $xy = 1$ and $xy = 2$ (cf. Fig. 2). It yields $xy \geq 1$ and $x, y > 0$.

To calculate the integral we define the change of variable $H\colon D \to \widetilde{D}$, where $(u, v) = H(x, y)$ with

$$\begin{cases} u = xy = H_1(x,y) \\ v = \frac{y}{x} = H_2(x,y) \end{cases}$$

and $x, y > 0$. By the definition of D, $\widetilde{D} = [1, 2] \times [1, 4]$. The Jacobian matrix of H is

$$J_H(x,y) = \begin{pmatrix} \partial_x H_1(x,y) & \partial_y H_1(x,y) \\ \partial_x H_2(x,y) & \partial_y H_2(x,y) \end{pmatrix} = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}$$

and its determinant is $\det\big(J_H(x,y)\big) = 2y/x > 0$.

Let $G = H^{-1}\colon \widetilde{D} \to D$ be the inverse transformation such that $(x, y) = G(u, v)$. The determinant of the Jacobian of G is therefore

$$\det\big(J_G(u,v)\big) = \left[\frac{1}{\det\big(J_H(x,y)\big)}\right]_{(x,y)=G(u,v)} = \left[\frac{x}{2y}\right]_{(x,y)=G(u,v)} = \frac{1}{2v} \geq \frac{1}{2\cdot 4} > 0.$$

because $v = \frac{y}{x}$. By applying the change of variables formula to the open $\mathring{D}$ and $\mathring{\widetilde{D}}$, we find

$$\begin{aligned}
\iint_D x^2y^2\,dx\,dy &= \iint_{\mathring{D}} x^2y^2\,dx\,dy = \iint_{\mathring{\widetilde{D}}} u^2\frac{1}{2v}\,du\,dv = \int_1^4 \left(\int_1^2 u^2\frac{1}{2v}\,du\right)dv \\
&= \int_1^4 \frac{1}{2v}\left[\frac{1}{3}u^3\right]_{u=1}^{u=2} dv = \int_1^4 \frac{7}{6}\,\frac{1}{v}\,dv \\
&= \frac{7}{6}\Big[\ln(|v|)\Big]_1^4 = \frac{7}{6}\,\ln(4) = \frac{7}{3}\,\ln(2)\,.
\end{aligned}$$

iii) We will use polar coordinates. Define $x = r\cos(\phi)$ and $y = r\sin(\phi)$ with $(r, \phi) \in \widetilde{D} = [1, 2] \times [\pi, 3\pi/2]$, we obtain

$$\iint\limits_D xy^3\,dx\,dy = \iint\limits_{\widetilde{D}} (r\cos\phi)(r\sin\phi)^3 r\,dr\,d\phi.$$

The final r factor comes from the Jacobian determinant of the coordinates change. Hence

$$\iint\limits_D xy^3\,dx\,dy = \int_1^2 \left(\int_\pi^{3\pi/2} r^5 \sin^3(\phi)\cos(\phi)\,d\phi \right) dr = \int_1^2 \left(r^5 \frac{1}{4}\sin^4(\phi) \right) \Big|_{\phi=\pi}^{\phi=3\pi/2} dr$$

$$= \int_1^2 \frac{1}{4} r^5\,dr = \frac{1}{24} r^6 \Big|_1^2 = \frac{2^6-1}{24} = \frac{21}{8}\,.$$

3.

i) Define $x = r\cos\theta,\ y = r\sin\theta$ and since the initial region is given by the equations

$$\begin{cases} x \geq 0 \\ y \leq 0 \\ x^2+y^2 = 9 \end{cases}$$

then the new region in polar coordinates is given by the equations

$$\begin{cases} \frac{3\pi}{2} \leq \theta \leq 2\pi \\ 0 \leq r \leq 3. \end{cases}$$

Therefore, we have

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2} \underset{\cdot}{y}\underset{\cdot}{x} = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^3 r e^{r^2} \underset{\cdot}{r}\underset{\cdot}{\theta} = \frac{\pi}{2} \cdot \frac{e^{r^2}}{2} \Big|_0^3 = \frac{\pi(e^9-1)}{4}.$$

ii) Define $x = r\cos\theta,\ y = r\sin\theta$ and since $D = \{(x,y) \in \mathbb{R}^2 \colon x, y \geq 0,\ x^2+y^2 \leq 2\}$, then the new region in polar coordinates is $\widetilde{D} = \{(\theta, r) \colon 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq \sqrt{2}\}$. Then we have

$$\iint\limits_D (4xy-7)\underset{\cdot}{x}\underset{\cdot}{y} = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (4r^2\cos\theta\sin\theta - 7)r\underset{\cdot}{r}\underset{\cdot}{\theta} = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (4r^3\cos\theta\sin\theta - 7r)\underset{\cdot}{r}\underset{\cdot}{\theta}$$

$$= \int_0^{\frac{\pi}{2}} \left(r^4\cos\theta\sin\theta - \frac{7}{2}r^2 \right) \Big|_0^{\sqrt{2}} \underset{\cdot}{\theta} = \int_0^{\frac{\pi}{2}} (4\cos\theta\sin\theta - 7)\underset{\cdot}{\theta}$$

$$= (-2\cos^2\theta - 7\theta)\Big|_0^{\frac{\pi}{2}} = 2 - \frac{7\pi}{2}.$$

4.

i) We have

$$\int_0^1 \int_0^{z^2} \int_0^{\sqrt{\frac{\pi}{2}}} xy\cos(x^2)\underset{\cdot}{x}\underset{\cdot}{y}\underset{\cdot}{z} = \int_0^1 \int_0^{z^2} y \left(\frac{\sin(x^2)}{2} \right) \Big|_0^{\sqrt{\frac{\pi}{2}}} \underset{\cdot}{y}\underset{\cdot}{z} = \frac{1}{2} \int_0^1 \int_0^{z^2} y\underset{\cdot}{y}\underset{\cdot}{z}$$

$$= \frac{1}{2} \int_0^1 \frac{y^2}{2} \Big|_0^{z^2} \underset{\cdot}{z} = \frac{1}{4} \int_0^1 z^4 \underset{\cdot}{z} = \frac{1}{4} \cdot \frac{z^5}{5} \Big|_0^1 = \frac{1}{20}.$$

ii) Since E is the region below $4x+y+2z=10$ in the first octant, we have that $0 \leq z \leq 5-2x-\frac{y}{2}$, and then $4x+y \leq 10$ with $x, y \geq 0$, thus $0 \leq y \leq 10-4x$, and then $4x \leq 10$, thus $0 \leq x \leq \frac{5}{2}$. Therefore, we have

$$\iiint\limits_E 6z^2 \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}} = \int_0^{\frac{5}{2}} \int_0^{10-4x} \int_0^{5-2x-\frac{y}{2}} 6z^2 \underset{\cdot}{\mathrm{z}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{x}}$$

$$= 2\int_0^{\frac{5}{2}} \int_0^{10-4x} \int_0^{5-2x-\frac{y}{2}} z^3 \Big|_0^{5-2x-\frac{y}{2}} \underset{\cdot}{\mathrm{z}}\underset{\cdot}{\mathrm{y}}$$

$$= 2\int_0^{\frac{5}{2}} \int_0^{10-4x} \left(5-2x-\frac{y}{2}\right)^3 \underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{x}}$$

$$= -\int_0^{\frac{5}{2}} \left(5-2x-\frac{y}{2}\right)^4 \Big|_0^{10-4x} \underset{\cdot}{\mathrm{x}}$$

$$= \int_0^{\frac{5}{2}} (5-2x)^4 \underset{\cdot}{\mathrm{x}} = -\frac{1}{10}(5-2x)^5 \Big|_0^{\frac{5}{2}} = \frac{625}{2}.$$

iii) Since E is the region bounded by $x = 2y^2+2z^2-5$ and $x = 1$, then $2y^2+2z^2-5 \leq x \leq 1$. The intersection of these two curves is the circle $y^2+z^2 = 3$, thus $-\sqrt{3-z^2} \leq y \leq \sqrt{3-z^2}$ and $-\sqrt{3} \leq z \leq \sqrt{3}$. Therefore, we have

$$\iiint\limits_E yz \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}} = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-z^2}}^{\sqrt{3-z^2}} \int_{2y^2+2z^2-5}^{1} yz \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}}$$

$$= 2\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-z^2}}^{\sqrt{3-z^2}} (3-y^2-z^2) yz \underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}}.$$

At this point it is more convenient to use polar coordinates so we set $y = r\sin\theta,\ z = r\cos\theta$ and then we have

$$\iiint\limits_E yz \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}} = 2\int_0^{2\pi} \int_0^{\sqrt{3}} (3-r^2)(r\sin\theta)(r\cos\theta) r \underset{\cdot}{\mathrm{r}}\underset{\cdot}{\theta}$$

$$= 2\left(\int_0^{2\pi} \sin\theta\cos\theta \underset{\cdot}{\theta}\right)\left(\int_0^{\sqrt{3}} (3r^3-r^5)\underset{\cdot}{\mathrm{r}}\right) = 0,$$

since $\int_0^{2\pi} \sin\theta\cos\theta\underset{\cdot}{\theta} = \frac{\sin^2\theta}{2}\Big|_0^{2\pi} = 0$.

5.

i) We set $u = \frac{2x-y}{2},\ v = \frac{y}{2},\ w = \frac{z}{3}$. Equivalently, we have $x = u+v,\ y = 2v,\ z = 3w$. Then

$$\frac{\partial x}{\partial u} = 1,\ \frac{\partial x}{\partial v} = 1,\ \frac{\partial y}{\partial v} = 2,\ \frac{\partial z}{\partial w} = 3$$

and all the other partial derivatives are 0. Then the Jacobian is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

with determinant 6. Thus we have

$$\int_0^3 \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(x + \frac{z}{3}\right) \underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}} = \int_0^1 \int_0^2 \int_0^1 6(u+v+w)\underset{\cdot}{\mathrm{u}}\underset{\cdot}{\mathrm{v}}\underset{\cdot}{\mathrm{w}}$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + v + w\right)\underset{\cdot}{\mathrm{v}}\underset{\cdot}{\mathrm{w}} = 6 \int_0^1 (3+2w)\underset{\cdot}{\mathrm{w}} = 24.$$

ii) We set $u = x,\ v = xy,\ w = 3z$, or equivalently, $x = u,\ y = \frac{v}{u},\ z = \frac{w}{3}$. Then the Jacobian is

$$\begin{pmatrix} 1 & 0 & 0 \\ -\dfrac{v}{u^2} & \dfrac{1}{u} & 0 \\ 0 & 0 & \dfrac{1}{3} \end{pmatrix}$$

with determinant $\frac{1}{3u}$. Then we have

$$\iiint_E (x^2y + 3xyz)\underset{\cdot}{\mathrm{x}}\underset{\cdot}{\mathrm{y}}\underset{\cdot}{\mathrm{z}} = \int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) \underset{\cdot}{\mathrm{u}}\underset{\cdot}{\mathrm{v}}\underset{\cdot}{\mathrm{w}}$$

$$= \int_0^3 \int_0^2 \left(\frac{v}{3} + \frac{(\log 2)vw}{3}\right) \underset{\cdot}{\mathrm{v}}\underset{\cdot}{\mathrm{w}} = \int_0^3 \left(\frac{2}{3} + \frac{2(\log 2)w}{3}\right) \underset{\cdot}{\mathrm{w}}$$

$$= 2 + \frac{2\log 2}{3} \cdot \frac{9}{2} = 2 + \log 8.$$