Duration : 144 minutes

EPFL

Analysis II
Exam
Common part
Spring 2 024

Answers

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = xy$$

and let $g(x,y) = x^2 + xy + y^2 - 12$. Then, under the constraint $g(x,y) = 0$, the maximal value of f is

- [ ] 6
- [ ] 8
- [x] 4
- [ ] -12

Question 2 : Let D be the subset of $\mathbb{R}^3$ given by

$$D = \left\{(x,y,z) \in \mathbb{R}^3 : y \geq 0,\ 1 \leq x^2 + y^2 + z^2 \leq 16\,,\ z \geq \tfrac{1}{2}\sqrt{x^2+y^2+z^2}\right\}.$$

Then the volume of D equals

- [ ] $\dfrac{\pi}{2}$
- [ ] $\dfrac{3\pi}{2}$
- [x] $\dfrac{21\pi}{2}$
- [ ] 63π

Question 3 : The subset

$$E = \left\{(x,y) \in \mathbb{R}^2 : x > 1 \text{ and } -1 < xy < 1\right\} \subset \mathbb{R}^2$$

- [x] is open and not bounded
- [ ] is closed and not bounded
- [ ] is closed and bounded
- [ ] is open and bounded

Question 4 : The solution $y(x)$ of the differential equation

$$y'(x) = \frac{x}{x^2+9}\big(y(x) - 1\big)$$

with initial condition $y(0) = 7$ also satisfies

- [ ] $y(4) = 0$
- [ ] $y(4) = 6$
- [ ] $y(4) = 26$
- [x] $y(4) = 11$

Question 5: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \ln(1+y^2) - xy\,.$$

Then

□ the function f admits exactly one local maximum in $\mathbb{R}^2$

□ the function f does not have stationary points in $\mathbb{R}^2$

■ the function f admits a stationary point in $\mathbb{R}^2$ that is not a local extremum

□ the function f admits exactly one local minimum in $\mathbb{R}^2$

Question 6: Let D be the subset of $\mathbb{R}^2$ given by

$$D = \big\{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2\,,\ x \ge 0\,,\ -x \le y \le x\big\}\,.$$

The integral

$$\iint_D (x^2 - y^2)\,dx\,dy$$

equals

■ $\frac{3}{4}$

□ $\frac{7}{3}$

□ $\frac{2}{3}\sqrt{2} - \frac{1}{3}$

□ $\frac{15}{4}$

Question 7: Let $F\colon (0,+\infty) \to \mathbb{R}$ be the function

$$F(t) = \int_0^t \ln(3e^{\,t-x} - 1)\,dx\,.$$

Then, for all $t > 0$,

■ $F'(t) = \ln(3e^{\,t} - 1)$

□ $F'(t) = \ln(3e^{\,t} - 1) - \ln(2)$

□ $F'(t) = \dfrac{3e^{\,t-x}}{3e^{\,t-x} - 1}$

□ $F'(t) = \ln(3e^{\,t} - 1) + \ln(2)$

Question 8: Let $D = \{(x, y, z) \in \mathbb{R}^3 : 0 < y < \pi\}$ and let $f : D \to \mathbb{R}$ be the function defined by

$$f(x, y, z) = \frac{x + z}{\sin(y)} + xyz\,.$$

The equation $f(x, y, z) = 2 + \frac{2}{\pi}$ implicitly defines a function $z = g(x, y)$ which satisfies $g\left(\frac{2}{\pi}, \frac{\pi}{2}\right) = 1$ and $f\big(x, y, g(x, y)\big) = 2 + \frac{2}{\pi}$ in a neighbourhood of $(x, y) = \left(\frac{2}{\pi}, \frac{\pi}{2}\right)$. The value of $\frac{\partial g}{\partial y}\left(\frac{2}{\pi}, \frac{\pi}{2}\right)$ is then

- ☐ $\frac{\partial g}{\partial y}\left(\frac{2}{\pi}, \frac{\pi}{2}\right) = -\frac{2}{\pi}$
- ☐ $\frac{\partial g}{\partial y}\left(\frac{2}{\pi}, \frac{\pi}{2}\right) = -\frac{\pi}{2}$
- ■ $\frac{\partial g}{\partial y}\left(\frac{2}{\pi}, \frac{\pi}{2}\right) = -\frac{1}{\pi}$
- ☐ $\frac{\partial g}{\partial y}\left(\frac{2}{\pi}, \frac{\pi}{2}\right) = -\frac{1}{2}$

Question 9: Let S be the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = x^4 y^3 + \cos(1 - x^2 y)\}$$

and let $z_0 \in \mathbb{R}$ be such that $(-1, 1, z_0) \in S$. Then the equation of the tangent plane to S at the point $(-1, 1, z_0)$ is

- ☐ $4x - 3y + z - 9 = 0$
- ■ $4x - 3y + z + 5 = 0$
- ☐ $4x - 3y + z - 2 = 0$
- ☐ $4x - 3y + z + 7 = 0$

Question 10: Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{f}(x, y) = \big(xy,\, x^2 + y^2\big)^{\mathrm{T}}$$

and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1 such that

$$\nabla g(1, 2) = \left(1,\, -\tfrac{1}{4}\right)^{\mathrm{T}}.$$

Then the composition $h = g \circ \boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}$ satisfies

- ☐ $\nabla h(1, 1) = \left(\frac{3}{2}, 0\right)^{\mathrm{T}}$
- ☐ $\nabla h(1, 1) = \left(\frac{7}{4}, 1\right)^{\mathrm{T}}$
- ☐ $\nabla h(1, 1) = \left(\frac{3}{4}, \frac{3}{2}\right)^{\mathrm{T}}$
- ■ $\nabla h(1, 1) = \left(\frac{1}{2}, \frac{1}{2}\right)^{\mathrm{T}}$

Question 11: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \sin(\pi x^2 y)\,.$$

Then the directional derivative $\nabla_{\boldsymbol{v}} f(1,1)$ of f at $(1,1)$ in the direction of the vector $\boldsymbol{v} = \left(-\frac{1}{\sqrt{5}}\,,\,\frac{2}{\sqrt{5}}\right)^{\mathrm{T}}$ is

- ☐ $-\dfrac{3\pi}{\sqrt{2}}$
- ■ 0
- ☐ -3π
- ☐ $\dfrac{6\pi}{\sqrt{5}}$

Question 12: The solution $y(x)$ of the differential equation

$$y'(x) + \frac{1}{x}\,y(x) = \frac{2}{(1+x^2)^2}$$

on the interval $(0,+\infty)$ with initial condition $y(1)=0$ also satisfies

- ■ $y(2) = \dfrac{3}{20}$
- ☐ $y(2) = -\dfrac{2}{15}$
- ☐ $y(2) = -\dfrac{1}{10}$
- ☐ $y(2) = 0$

Question 13: Let $(\boldsymbol{x}_n)$ be the sequence of elements in $\mathbb{R}^2$ defined by

$$\boldsymbol{x}_n = \left(n\sin\Bigl(\frac{(-1)^n}{n}\Bigr),\ \frac{(-1)^n}{n}\,\sin(n)\right)^{\mathrm{T}}, \qquad \text{for all } n \in \{1,2,3,\ldots\}.$$

Then

- ■ the sequence is bounded but does not converge
- ☐ the sequence converges and $\lim_{n\to\infty} \boldsymbol{x}_n = (0,\,1)^{\mathrm{T}}$
- ☐ the sequence is not bounded
- ☐ the sequence converges and $\lim_{n\to\infty} \boldsymbol{x}_n = (1,\,0)^{\mathrm{T}}$

Question 14: Let $f\colon \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ be the function defined by

$$f(x,y) = \frac{1-\cos(x)}{x^2+y^2}\,.$$

Then

- [ ] $\lim\limits_{(x,y)\to(0,0)} f(x,y) = -\dfrac{1}{2}$
- [x] $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist
- [ ] $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{1}{2}$
- [ ] $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

Question 15: The solution $y(x)$ of the differential equation

$$y''(x) - y'(x) - 6y(x) = 4e^{-x}$$

with initial conditions $y(0) = -2$ and $y'(0) = 3$ also satisfies

- [ ] $y\big(\ln(3)\big) = -12$
- [ ] $y\big(\ln(3)\big) = 0$
- [ ] $y\big(\ln(3)\big) = -\dfrac{1}{3}$
- [x] $y\big(\ln(3)\big) = -\dfrac{4}{9}$

Question 16: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3y+y^3x}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- [x] f is differentiable at $(0,0)$
- [ ] $\dfrac{\partial f}{\partial x}(0,0)$ does not exist
- [ ] f is not continuous at $(0,0)$
- [ ] f is continuous at $(0,0)$, but f is not differentiable at $(0,0)$

Question 17: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = 2\cos(3y - x) - \sin^2(4x - y)\,.$$

The second order Taylor polynomial of f at the point $(0,0)$ is

- [x] $p_2(x,y) = 2 - 17x^2 + 14xy - 10y^2$
- [ ] $p_2(x,y) = 2 - 4x + y - 2x^2 + 12xy - 18y^2$
- [ ] $p_2(x,y) = 2 - 18x^2 + 20xy - 19y^2$
- [ ] $p_2(x,y) = 2 - 4x + y - x^2 + 6xy - 9y^2$

Question 18: Let

$$I = \int_0^1 \left(\int_{\sqrt{x}}^1 \frac{5x}{1+2y^5}\, dy \right) dx\,.$$

Then

- [ ] $I = \ln(2)$
- [x] $I = \frac{1}{4}\ln(3)$
- [ ] $I = \frac{1}{2}\ln(3)$
- [ ] $I = \frac{1}{2}\ln(2)$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. If $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ are two functions such that $\lim_{r\to 0} g(r) = 0$, $\lim_{r\to 0} h(r) = 0$ and

$$g(r) \le f\big(r\cos(\theta)\,,\, r\sin(\theta)\big) \le h(r) \qquad \text{for all } r > 0 \text{ and for all } \theta \in \mathbb{R},$$

then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

■ TRUE ☐ FALSE

Question 20 : Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function that is differentiable at $(0,0)$, and suppose

$$f(0,0) = 0 \quad \text{and} \quad \nabla f(0,0) = \big(0,\, 0\big)^{\mathrm{T}}.$$

Then

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0.$$

■ TRUE ☐ FALSE

Question 21 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$g(x,y) = f(x^2+y^2)\,.$$

If $D = \big\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\big\}$, then

$$\iint_D g(x,y)\,dx\,dy = 2\pi \int_0^1 f(r^2)\, r\, dr\,.$$

■ TRUE ☐ FALSE

Question 22 : Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^2 and let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$g(x,y) = \big(f(x,y)\big)^2 \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

If $p(x,y)$ is the first order Taylor polynomial of f at the point $(0,0)$, then $\big(p(x,y)\big)^2$ is the second order Taylor polynomial of g at the point $(0,0)$.

☐ TRUE ■ FALSE

Question 23 : Let $f\colon U \to \mathbb{R}$ be a function defined on a nonempty open set $U \subset \mathbb{R}^2$.
If $(x_0,y_0) \in U$ is such that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, then $f(x_0,y_0) = L$.

☐ TRUE ■ FALSE

Question 24 : Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be two functions of class C^1. If (x_0, y_0) is an extremum of f under the constraint $g(x, y) = 0$, then $\nabla f(x_0, y_0) = \big(0,\, 0\big)^{\mathrm{T}}$.

□ TRUE ■ FALSE

Question 25 : Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. If $(x_0, y_0) \in \mathbb{R}^2$ is such that $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, then $\nabla_{\boldsymbol{v}} f(x_0, y_0) = 0$ for all $\boldsymbol{v} \in \mathbb{R}^2$ with $\|\boldsymbol{v}\| = 1$.

□ TRUE ■ FALSE

Question 26 : If $A \subset \mathbb{R}^2$ is a nonempty open subset of $\mathbb{R}^2$, then the set

$$\big\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in A \text{ and } z = 0\big\}$$

is an open subset of $\mathbb{R}^3$.

□ TRUE ■ FALSE

Question 27 : Let A and B be two nonempty subsets of $\mathbb{R}^n$. Then

$$\partial(A \cup B) = (\partial A) \cup (\partial B)$$

□ TRUE ■ FALSE

Question 28 : Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1. If $z = 3$ is the equation of the tangent plane to the graph of f at the point $(2, 0, 3)$, then $\nabla f(2, 0) = \big(0,\, 0\big)^{\mathrm{T}}$.

■ TRUE □ FALSE