Duration : 144 minutes

# Analysis II
# Exam
## Common part
## Spring 2 024

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## Questions

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For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

$0$ points if you give no answer or more than one,

$-1$ if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

$0$ points if you give no answer or more than one,

$-1$ points if your answer is incorrect.

## First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

**Question 1:** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = xy$$

and let $g(x,y) = x^2 + xy + y^2 - 12$. Then, under the constraint $g(x,y) = 0$, the maximal value of $f$ is

- ☐ 6
- ☐ 8
- ☐ 4
- ☐ $-12$

**Question 2:** Let $D$ be the subset of $\mathbb{R}^3$ given by

$$D = \Big\{(x,y,z) \in \mathbb{R}^3 : y \geq 0,\ 1 \leq x^2 + y^2 + z^2 \leq 16\,,\ z \geq \tfrac{1}{2}\sqrt{x^2+y^2+z^2}\Big\}.$$

Then the volume of $D$ equals

- ☐ $\dfrac{\pi}{2}$
- ☐ $\dfrac{3\pi}{2}$
- ☐ $\dfrac{21\pi}{2}$
- ☐ $63\pi$

**Question 3:** The subset

$$E = \big\{(x,y) \in \mathbb{R}^2 : x > 1 \text{ and } -1 < xy < 1\big\} \subset \mathbb{R}^2$$

- ☐ is open and not bounded
- ☐ is closed and not bounded
- ☐ is closed and bounded
- ☐ is open and bounded

**Question 4:** The solution $y(x)$ of the differential equation

$$y'(x) = \frac{x}{x^2+9}\big(y(x) - 1\big)$$

with initial condition $y(0) = 7$ also satisfies

- ☐ $y(4) = 0$
- ☐ $y(4) = 6$
- ☐ $y(4) = 26$
- ☐ $y(4) = 11$

**Question 5:** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \ln(1+y^2) - xy\,.$$

Then

- ☐ the function $f$ admits exactly one local maximum in $\mathbb{R}^2$
- ☐ the function $f$ does not have stationary points in $\mathbb{R}^2$
- ☐ the function $f$ admits a stationary point in $\mathbb{R}^2$ that is not a local extremum
- ☐ the function $f$ admits exactly one local minimum in $\mathbb{R}^2$

**Question 6:** Let $D$ be the subset of $\mathbb{R}^2$ given by

$$D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2\,,\ x \ge 0\,,\ -x \le y \le x\}\,.$$

The integral

$$\iint_D (x^2 - y^2)\,dx\,dy$$

equals

- ☐ $\frac{3}{4}$
- ☐ $\frac{7}{3}$
- ☐ $\frac{2}{3}\sqrt{2} - \frac{1}{3}$
- ☐ $\frac{15}{4}$

**Question 7:** Let $F\colon (0,+\infty) \to \mathbb{R}$ be the function

$$F(t) = \int_0^t \ln(3e^{\,t-x} - 1)\,dx\,.$$

Then, for all $t > 0$,

- ☐ $F'(t) = \ln(3e^{\,t} - 1)$
- ☐ $F'(t) = \ln(3e^{\,t} - 1) - \ln(2)$
- ☐ $F'(t) = \dfrac{3e^{\,t-x}}{3e^{\,t-x} - 1}$
- ☐ $F'(t) = \ln(3e^{\,t} - 1) + \ln(2)$

**Question 8 :** Let $D = \{(x,y,z) \in \mathbb{R}^3 : 0 < y < \pi\}$ and let $f\colon D \to \mathbb{R}$ be the function defined by

$$f(x,y,z) = \frac{x+z}{\sin(y)} + xyz\,.$$

The equation $f(x,y,z) = 2+\frac{2}{\pi}$ implicitly defines a function $z = g(x,y)$ which satisfies $g\left(\frac{2}{\pi}, \frac{\pi}{2}\right) = 1$ and $f\big(x,y,g(x,y)\big) = 2+\frac{2}{\pi}$ in a neighbourhood of $(x,y) = \left(\frac{2}{\pi}, \frac{\pi}{2}\right)$. The value of $\frac{\partial g}{\partial y}\left(\frac{2}{\pi}, \frac{\pi}{2}\right)$ is then

- ☐ $\dfrac{\partial g}{\partial y}\left(\dfrac{2}{\pi}, \dfrac{\pi}{2}\right) = -\dfrac{2}{\pi}$
- ☐ $\dfrac{\partial g}{\partial y}\left(\dfrac{2}{\pi}, \dfrac{\pi}{2}\right) = -\dfrac{\pi}{2}$
- ☐ $\dfrac{\partial g}{\partial y}\left(\dfrac{2}{\pi}, \dfrac{\pi}{2}\right) = -\dfrac{1}{\pi}$
- ☐ $\dfrac{\partial g}{\partial y}\left(\dfrac{2}{\pi}, \dfrac{\pi}{2}\right) = -\dfrac{1}{2}$

**Question 9 :** Let $S$ be the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = x^4y^3 + \cos(1 - x^2y)\}$$

and let $z_0 \in \mathbb{R}$ be such that $(-1,1,z_0) \in S$. Then the equation of the tangent plane to $S$ at the point $(-1,1,z_0)$ is

- ☐ $4x - 3y + z - 9 = 0$
- ☐ $4x - 3y + z + 5 = 0$
- ☐ $4x - 3y + z - 2 = 0$
- ☐ $4x - 3y + z + 7 = 0$

**Question 10 :** Let $\boldsymbol{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{f}(x,y) = \big(xy,\, x^2 + y^2\big)^{\mathrm{T}}$$

and let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^1$ such that

$$\nabla g(1,2) = \left(1,\, -\tfrac{1}{4}\right)^{\mathrm{T}}.$$

Then the composition $h = g \circ \boldsymbol{f}\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies

- ☐ $\nabla h(1,1) = \left(\frac{3}{2}, 0\right)^{\mathrm{T}}$
- ☐ $\nabla h(1,1) = \left(\frac{7}{4}, 1\right)^{\mathrm{T}}$
- ☐ $\nabla h(1,1) = \left(\frac{3}{4}, \frac{3}{2}\right)^{\mathrm{T}}$
- ☐ $\nabla h(1,1) = \left(\frac{1}{2}, \frac{1}{2}\right)^{\mathrm{T}}$

**Question 11 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \sin(\pi x^2 y)\,.$$

Then the directional derivative $\nabla_{\boldsymbol{v}} f(1,1)$ of $f$ at $(1,1)$ in the direction of the vector $\boldsymbol{v} = \left(-\frac{1}{\sqrt{5}}\,,\,\frac{2}{\sqrt{5}}\right)^{\mathrm{T}}$ is

- ☐ $-\dfrac{3\pi}{\sqrt{2}}$
- ☐ $0$
- ☐ $-3\pi$
- ☐ $\dfrac{6\pi}{\sqrt{5}}$

**Question 12 :** The solution $y(x)$ of the differential equation

$$y'(x) + \frac{1}{x}\,y(x) = \frac{2}{(1+x^2)^2}$$

on the interval $(0,+\infty)$ with initial condition $y(1) = 0$ also satisfies

- ☐ $y(2) = \dfrac{3}{20}$
- ☐ $y(2) = -\dfrac{2}{15}$
- ☐ $y(2) = -\dfrac{1}{10}$
- ☐ $y(2) = 0$

**Question 13 :** Let $\left(\boldsymbol{x}_n\right)$ be the sequence of elements in $\mathbb{R}^2$ defined by

$$\boldsymbol{x}_n = \left(n\sin\!\left(\frac{(-1)^n}{n}\right),\ \frac{(-1)^n}{n}\sin(n)\right)^{\mathrm{T}}, \qquad \text{for all } n \in \{1,2,3,\ldots\}.$$

Then

- ☐ the sequence is bounded but does not converge
- ☐ the sequence converges and $\lim\limits_{n\to\infty} \boldsymbol{x}_n = \left(0,\,1\right)^{\mathrm{T}}$
- ☐ the sequence is not bounded
- ☐ the sequence converges and $\lim\limits_{n\to\infty} \boldsymbol{x}_n = \left(1,\,0\right)^{\mathrm{T}}$

**Question 14 :** Let $f\colon \mathbb{R}^2 \backslash \big\{(0,0)\big\} \to \mathbb{R}$ be the function defined by

$$f(x,y) = \frac{1-\cos(x)}{x^2+y^2}\,.$$

Then

- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = -\frac{1}{2}$
- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist
- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$
- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

**Question 15 :** The solution $y(x)$ of the differential equation

$$y''(x) - y'(x) - 6y(x) = 4e^{-x}$$

with initial conditions $y(0) = -2$ and $y'(0) = 3$ also satisfies

- ☐ $y\big(\ln(3)\big) = -12$
- ☐ $y\big(\ln(3)\big) = 0$
- ☐ $y\big(\ln(3)\big) = -\frac{1}{3}$
- ☐ $y\big(\ln(3)\big) = -\frac{4}{9}$

**Question 16 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3y+y^3x}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- ☐ $f$ is differentiable at $(0,0)$
- ☐ $\frac{\partial f}{\partial x}(0,0)$ does not exist
- ☐ $f$ is not continuous at $(0,0)$
- ☐ $f$ is continuous at $(0,0)$, but $f$ is not differentiable at $(0,0)$

**Question 17 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = 2\cos(3y - x) - \sin^2(4x - y)\,.$$

The second order Taylor polynomial of $f$ at the point $(0,0)$ is

- ☐ $p_2(x,y) = 2 - 17x^2 + 14xy - 10y^2$
- ☐ $p_2(x,y) = 2 - 4x + y - 2x^2 + 12xy - 18y^2$
- ☐ $p_2(x,y) = 2 - 18x^2 + 20xy - 19y^2$
- ☐ $p_2(x,y) = 2 - 4x + y - x^2 + 6xy - 9y^2$

**Question 18 :** Let

$$I = \int_0^1 \left( \int_{\sqrt{x}}^1 \frac{5x}{1+2y^5}\, dy \right) dx\,.$$

Then

- ☐ $I = \ln(2)$
- ☐ $I = \frac{1}{4}\ln(3)$
- ☐ $I = \frac{1}{2}\ln(3)$
- ☐ $I = \frac{1}{2}\ln(2)$

## Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

**Question 19 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. If $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ are two functions such that $\lim_{r\to 0} g(r) = 0$, $\lim_{r\to 0} h(r) = 0$ and

$$g(r) \le f\big(r\cos(\theta)\,,\, r\sin(\theta)\big) \le h(r) \qquad \text{for all } r > 0 \text{ and for all } \theta \in \mathbb{R},$$

then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

☐ TRUE ☐ FALSE

**Question 20 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function that is differentiable at $(0,0)$, and suppose

$$f(0,0) = 0 \quad \text{and} \quad \nabla f(0,0) = \big(0,\,0\big)^{\mathrm{T}}.$$

Then

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0.$$

☐ TRUE ☐ FALSE

**Question 21 :** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$g(x,y) = f(x^2+y^2)\,.$$

If $D = \big\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\big\}$, then

$$\iint_D g(x,y)\,dx\,dy = 2\pi \int_0^1 f(r^2)\,r\,dr\,.$$

☐ TRUE ☐ FALSE

**Question 22 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^2$ and let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$g(x,y) = \big(f(x,y)\big)^2 \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

If $p(x,y)$ is the first order Taylor polynomial of $f$ at the point $(0,0)$, then $\big(p(x,y)\big)^2$ is the second order Taylor polynomial of $g$ at the point $(0,0)$.

☐ TRUE ☐ FALSE

**Question 23 :** Let $f\colon U \to \mathbb{R}$ be a function defined on a nonempty open set $U \subset \mathbb{R}^2$.
If $(x_0,y_0) \in U$ is such that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, then $f(x_0,y_0) = L$.

☐ TRUE ☐ FALSE

**Question 24 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be two functions of class $C^1$. If $(x_0, y_0)$ is an extremum of $f$ under the constraint $g(x, y) = 0$, then $\nabla f(x_0, y_0) = \big(0,\, 0\big)^{\mathrm{T}}$.

☐ TRUE ☐ FALSE

**Question 25 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. If $(x_0, y_0) \in \mathbb{R}^2$ is such that $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, then $\nabla_{\boldsymbol{v}} f(x_0, y_0) = 0$ for all $\boldsymbol{v} \in \mathbb{R}^2$ with $\|\boldsymbol{v}\| = 1$.

☐ TRUE ☐ FALSE

**Question 26 :** If $A \subset \mathbb{R}^2$ is a nonempty open subset of $\mathbb{R}^2$, then the set

$$\big\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in A \text{ and } z = 0\big\}$$

is an open subset of $\mathbb{R}^3$.

☐ TRUE ☐ FALSE

**Question 27 :** Let $A$ and $B$ be two nonempty subsets of $\mathbb{R}^n$. Then

$$\partial(A \cup B) = (\partial A) \cup (\partial B)$$

☐ TRUE ☐ FALSE

**Question 28 :** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^1$. If $z = 3$ is the equation of the tangent plane to the graph of $f$ at the point $(2, 0, 3)$, then $\nabla f(2, 0) = \big(0,\, 0\big)^{\mathrm{T}}$.

☐ TRUE ☐ FALSE