Duration : 144 minutes

EPFL

Analysis II Exam

Common part

Spring 2023

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = xy^2 - x^2 y\,.$$

Then the directional derivative of f at $(2,1)$ in the direction of the vector $\boldsymbol{v} = \left(\frac{3}{5}\,,\,-\frac{4}{5}\right)^{\mathrm{T}}$ is

- ☐ $-\dfrac{2}{5}$
- ☐ $-\dfrac{1}{5}$
- ■ $-\dfrac{9}{5}$
- ☐ $-\dfrac{8}{5}$

Question 2 : Let D be the subset of $\mathbb{R}^3$ given by

$$D = \left\{\, (x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ 4x + 2y + z \leq 8 \,\right\}.$$

The volume of D is

- ■ $\displaystyle\int_0^2 \left(\int_0^{4-2x} \left(\int_0^{8-4x-2y} 1\; dz \right) dy \right) dx$
- ☐ $\displaystyle\int_0^4 \left(\int_0^{8-2y} \left(\int_0^{8-4x} 1\; dz \right) dx \right) dy$
- ☐ $\displaystyle\int_0^2 \left(\int_0^{2-4x} \left(\int_0^{8-2y} 1\; dz \right) dy \right) dx$
- ☐ $\displaystyle\int_0^8 \left(\int_0^{2-y} \left(\int_0^{8-4x-2y} 1\; dz \right) dx \right) dy$

Question 3 : The solution $y(x)$ of the differential equation

$$x\,y'(x) = \frac{\big(y(x)\big)^2}{\ln(x)+1}$$

on the interval $(1, +\infty)$ with initial condition $y(e) = -\dfrac{1}{2\ln(2)}$ also satisfies

- ☐ $y(e^3) = \dfrac{1}{4\ln(2)}$
- ■ $y(e^3) = -\dfrac{1}{3\ln(2)}$
- ☐ $y(e^3) = -\dfrac{1}{\ln(2)}$
- ☐ $y(e^3) = -\dfrac{1}{2\ln(2)}$

Question 4: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = e^{x^2+4x-2y}\,.$$

The second order Taylor polynomial for f around $(0,0)$ is

- ☐ $p_2(x,y) = 1 + 4x - 2y + 9x^2 - 4xy + 2y^2$
- ☐ $p_2(x,y) = 1 + 4x - 2y + 18x^2 - 8xy + 4y^2$
- ☐ $p_2(x,y) = 1 + 4x - 2y + 18x^2 - 16xy + 4y^2$
- ■ $p_2(x,y) = 1 + 4x - 2y + 9x^2 - 8xy + 2y^2$

Question 5: Let $F : (0,+\infty) \to \mathbb{R}$ be the function defined by

$$F(t) = \int_{1/t}^{t^2} \frac{\ln(1+tx)}{x}\,dx\,.$$

Then, for all $t > 0$,

- ☐ $F'(t) = \dfrac{2\ln(1+t^3)}{t} + \dfrac{\ln(2)}{t}$
- ■ $F'(t) = \dfrac{3\ln(1+t^3)}{t}$
- ☐ $F'(t) = \dfrac{3\ln(1+t^3)}{t} - \dfrac{2\ln(2)}{t}$
- ☐ $F'(t) = \dfrac{\ln(1+t^3)}{t^2} - t\ln(2) + \dfrac{\ln(1+t^3)}{t} - \dfrac{\ln(2)}{t}$

Question 6: Let

$$I = \int_0^1 \left(\int_{y^3}^1 \frac{36y^8}{1+x^4}\,dx \right) dy\,.$$

Then

- ☐ $I = \frac{1}{4}\ln(2)$
- ☐ $I = 2\ln(2)$
- ☐ $I = \frac{1}{2}\ln(2)$
- ■ $I = \ln(2)$

Question 7: Let $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{g}(x, y, z) = \big(xy,\ xz,\ yz\big)^{\mathrm{T}}$$

and let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(u, v, w) = uvw\,.$$

Then the composition $h = f \circ \boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}$ satisfies

- ☐ $\dfrac{\partial h}{\partial z}(-1, 0, 1) = 1$
- ■ $\dfrac{\partial h}{\partial z}(-1, 0, 1) = 0$
- ☐ $\dfrac{\partial h}{\partial z}(-1, 0, 1) = 2$
- ☐ $\dfrac{\partial h}{\partial z}(-1, 0, 1) = -1$

Question 8: Let S be the surface

$$S = \big\{(x, y, z) \in \mathbb{R}^3 : 3x + y^2 - e^z = 0\big\}$$

and let $z_0 \in \mathbb{R}$ be such that $(-1, 2, z_0) \in S$. Then

- ■ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x + 4y - z - 5 = 0$
- ☐ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x - 2y - z + 7 = 0$
- ☐ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x + 2y - z - 1 = 0$
- ☐ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x - 4y - z + 11 = 0$

Question 9: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(x, y, z) = 3x^2 e^z + ye^x + yz^2\,.$$

The equation $f(x, y, z) = 3$ implicitly defines a function $z = g(x, y)$ which satisfies $g(1, 0) = 0$ and $f\big(x, y, g(x, y)\big) = 3$ in a neighbourhood of $(x, y) = (1, 0)$. The value of $\frac{\partial g}{\partial y}(1, 0)$ is then

- ☐ $\dfrac{\partial g}{\partial y}(1, 0) = -3$
- ☐ $\dfrac{\partial g}{\partial y}(1, 0) = \dfrac{e}{2}$
- ☐ $\dfrac{\partial g}{\partial y}(1, 0) = -e$
- ■ $\dfrac{\partial g}{\partial y}(1, 0) = -\dfrac{e}{3}$

Question 10: The subset

$$A = \big\{(x, y) \in \mathbb{R}^2 : x^2 \le 1 \text{ and } y \le x^2\big\} \subset \mathbb{R}^2$$

- ☐ is not bounded and not closed
- ☐ is closed and bounded
- ☐ is bounded and not closed
- ■ is closed and not bounded

Question 11 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = e^{x-y} - x + e^{2y}\,.$$

Then

- ■ the function f admits exactly one local minimum in $\mathbb{R}^2$
- □ the function f admits exactly one local maximum in $\mathbb{R}^2$
- □ the function f admits a stationary point in $\mathbb{R}^2$ that is not a local extremum
- □ the function f does not have stationary points in $\mathbb{R}^2$

Question 12 : Let D be the subset of $\mathbb{R}^2$ given by

$$D = \big\{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4,\ y \ge |x|\big\}\,.$$

Then the integral

$$\iint_D y\,dx\,dy$$

equals

- □ $\dfrac{7\pi}{6}$
- ■ $\dfrac{7\sqrt{2}}{3}$
- □ 0
- □ $\dfrac{3\sqrt{2}}{2}$

Question 13 : Let $(\boldsymbol{a}_n)$ be the sequence of elements in $\mathbb{R}^3$ defined by

$$\boldsymbol{a}_n = \left(\frac{(-1)^n}{n},\ (-1)^n,\ (-1)^n n\right)^{\mathrm{T}}, \qquad \text{for all } n \in \{1, 2, 3, \ldots\}.$$

Then

- □ the sequence has a bounded subsequence
- □ the sequence is bounded
- □ the sequence has a converging subsequence
- ■ the limit $\lim_{n\to\infty} \boldsymbol{a}_n$ does not exist

Question 14 : Let $f : \mathbb{R}^2 \backslash \big\{(0, 0)\big\} \to \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{y^2 \sin^2(x)}{x^4 + y^2}\,.$$

Then

- □ $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = \frac{1}{2}$
- ■ $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = 0$
- □ $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist
- □ $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = \frac{\pi^2}{4}$

Question 15 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- ☐ f is not continuous at $(0,0)$
- ☐ f is continuous at $(0,0)$, but $\frac{\partial f}{\partial x}(0,0)$ does not exist
- ■ f is continuous at $(0,0)$, $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist, but f is not differentiable at $(0,0)$
- ☐ f is differentiable at $(0,0)$

Question 16 : The solution $y(x)$ of the differential equation

$$y'(x) + \frac{e^x}{e^x+1}\,y(x) = \frac{1}{e^x+1}$$

with initial condition $y(1) = 0$ also satisfies

- ☐ $y(0) = -2$
- ☐ $y(0) = \frac{1}{2}$
- ■ $y(0) = -\frac{1}{2}$
- ☐ $y(0) = 2$

Question 17 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = xy^3$$

and let $g(x,y) = x^2 + y^2 - 1$. Then, under the constraint $g(x,y) = 0$, the maximal value of f is

- ☐ $\frac{\sqrt{3}}{8}$
- ■ $\frac{3\sqrt{3}}{16}$
- ☐ $\frac{\sqrt{3}}{16}$
- ☐ $\frac{1}{4}$

Question 18 : The solution $y(x)$ of the differential equation

$$y''(x) - 3y'(x) + 2y(x) = 2x^2 - 2x$$

with initial conditions $y(0) = 2$ and $y'(0) = 2$ also satisfies

- ☐ $y(1) = \frac{5}{4} + 2e$
- ☐ $y(1) = 5 - 2e^2$
- ■ $y(1) = 5$
- ☐ $y(1) = 5 + 2e - 2e^2$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class C^3. Then

$$\frac{\partial^3 f}{\partial x^2 \partial y}(x, y, z) = \frac{\partial^3 f}{\partial x \partial y \partial x}(x, y, z)\,, \qquad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

■ TRUE ☐ FALSE

Question 20 : Let A and B be two nonempty subsets of $\mathbb{R}^2$. Then the subset C defined by

$$C = \overline{A} \cup \overline{B}$$

is a closed subset of $\mathbb{R}^2$.

■ TRUE ☐ FALSE

Question 21 : Let $f : \mathbb{R} \to (0, 1)$ be a function such that $\lim_{t \to 0} f(t) = L > 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{f\big(x^2 + y^2\big)} = 0\,.$$

■ TRUE ☐ FALSE

Question 22 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^2 and let $\boldsymbol{a} \in \mathbb{R}^2$ be a stationary point of f. If the two eigenvalues of the Hessian matrix $\mathrm{Hess}_f(\boldsymbol{a})$, λ_1 and λ_2, satisfy

$$\lambda_1 \lambda_2 > 0\,,$$

then $\boldsymbol{a}$ is a local extremum of f.

■ TRUE ☐ FALSE

Question 23 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^∞ and let p_1 et p_2 be its first and second order Taylor polynomials around $(x_0, y_0) = (0, 0)$. If p_1 is the zero polynomial, then p_2 is also the zero polynomial.

☐ TRUE ■ FALSE

Question 24 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, y) = \big(f(x, y)\big)^2$. If f is not differentiable at $(x, y) = (0, 0)$, then g is not differentiable at $(x, y) = (0, 0)$.

☐ TRUE ■ FALSE

Question 25 : Let $A = \mathbb{R}^3 \backslash \{(0, 0, 0)\}$. Then the boundary ∂A is the empty set.

☐ TRUE ■ FALSE

Question 26 : If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^2, then f admits a tangent plane at each point of its graph.

■ TRUE ☐ FALSE

Question 27 : Let D_1 and D_2 be the subsets of $\mathbb{R}^3$ given by

$$D_1 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 4\right\} \quad \text{and} \quad D_2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 9\right\}.$$

If $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, then

$$\iiint_{D_1} f(x,y,z)\, dx\, dy\, dz < \iiint_{D_2} f(x,y,z)\, dx\, dy\, dz\,.$$

☐ TRUE ■ FALSE

Question 28 : Let D be a nonempty, open, and bounded subset of $\mathbb{R}^n$. If $f : D \to \mathbb{R}$ is a continuous function, then f attains its global maximum on D.

☐ TRUE ■ FALSE