

Analysis II

Exam

Common part

Spring 2023

Questions

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = xy^2 - x^2y.$$

Then the directional derivative of f at $(2, 1)$ in the direction of the vector $\mathbf{v} = \left(\frac{3}{5}, -\frac{4}{5}\right)^T$ is

☐ $-\frac{2}{5}$

☐ $-\frac{1}{5}$

☐ $-\frac{9}{5}$

☐ $-\frac{8}{5}$

Question 2: Let D be the subset of $\mathbb{R}^3$ given by

$$D = \{ (x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, 4x + 2y + z \leq 8 \}.$$

The volume of D is

☐ $\int_0^2 \left(\int_0^{4-2x} \left(\int_0^{8-4x-2y} 1 \, dz \right) dy \right) dx$

☐ $\int_0^4 \left(\int_0^{8-2y} \left(\int_0^{8-4x} 1 \, dz \right) dx \right) dy$

☐ $\int_0^2 \left(\int_0^{2-4x} \left(\int_0^{8-2y} 1 \, dz \right) dy \right) dx$

☐ $\int_0^8 \left(\int_0^{2-y} \left(\int_0^{8-4x-2y} 1 \, dz \right) dx \right) dy$

Question 3: The solution $y(x)$ of the differential equation

$$x y'(x) = \frac{(y(x))^2}{\ln(x) + 1}$$

on the interval $(1, +\infty)$ with initial condition $y(e) = -\frac{1}{2 \ln(2)}$ also satisfies

☐ $y(e^3) = \frac{1}{4 \ln(2)}$

☐ $y(e^3) = -\frac{1}{3 \ln(2)}$

☐ $y(e^3) = -\frac{1}{\ln(2)}$

☐ $y(e^3) = -\frac{1}{2 \ln(2)}$

Question 4: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = e^{x^2+4x-2y}.$$

The second order Taylor polynomial for f around $(0, 0)$ is

☐ $p_2(x, y) = 1 + 4x - 2y + 9x^2 - 4xy + 2y^2$

☐ $p_2(x, y) = 1 + 4x - 2y + 18x^2 - 8xy + 4y^2$

☐ $p_2(x, y) = 1 + 4x - 2y + 18x^2 - 16xy + 4y^2$

☐ $p_2(x, y) = 1 + 4x - 2y + 9x^2 - 8xy + 2y^2$

Question 5: Let $F : (0, +\infty) \rightarrow \mathbb{R}$ be the function defined by

$$F(t) = \int_{1/t}^{t^2} \frac{\ln(1+tx)}{x} dx.$$

Then, for all $t > 0$,

☐ $F'(t) = \frac{2 \ln(1+t^3)}{t} + \frac{\ln(2)}{t}$

☐ $F'(t) = \frac{3 \ln(1+t^3)}{t}$

☐ $F'(t) = \frac{3 \ln(1+t^3)}{t} - \frac{2 \ln(2)}{t}$

☐ $F'(t) = \frac{\ln(1+t^3)}{t^2} - t \ln(2) + \frac{\ln(1+t^3)}{t} - \frac{\ln(2)}{t}$

Question 6: Let

$$I = \int_0^1 \left(\int_{y^3}^1 \frac{36y^8}{1+x^4} dx \right) dy.$$

Then

☐ $I = \frac{1}{4} \ln(2)$

☐ $I = 2 \ln(2)$

☐ $I = \frac{1}{2} \ln(2)$

☐ $I = \ln(2)$

Question 7: Let $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the function defined by

$$g(x, y, z) = (xy, xz, yz)^T$$

and let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the function defined by

$$f(u, v, w) = uvw.$$

Then the composition $h = f \circ g : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfies

☐ $\frac{\partial h}{\partial z}(-1, 0, 1) = 1$

☐ $\frac{\partial h}{\partial z}(-1, 0, 1) = 0$

☐ $\frac{\partial h}{\partial z}(-1, 0, 1) = 2$

☐ $\frac{\partial h}{\partial z}(-1, 0, 1) = -1$

Question 8: Let S be the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : 3x + y^2 - e^z = 0\}$$

and let $z_0 \in \mathbb{R}$ be such that $(-1, 2, z_0) \in S$. Then

☐ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x + 4y - z - 5 = 0$

☐ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x - 2y - z + 7 = 0$

☐ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x + 2y - z - 1 = 0$

☐ the equation of the tangent plane to S at the point $(-1, 2, z_0)$ is $3x - 4y - z + 11 = 0$

Question 9: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y, z) = 3x^2e^z + ye^x + yz^2.$$

The equation $f(x, y, z) = 3$ implicitly defines a function $z = g(x, y)$ which satisfies $g(1, 0) = 0$ and $f(x, y, g(x, y)) = 3$ in a neighbourhood of $(x, y) = (1, 0)$. The value of $\frac{\partial g}{\partial y}(1, 0)$ is then

☐ $\frac{\partial g}{\partial y}(1, 0) = -3$

☐ $\frac{\partial g}{\partial y}(1, 0) = \frac{e}{2}$

☐ $\frac{\partial g}{\partial y}(1, 0) = -e$

☐ $\frac{\partial g}{\partial y}(1, 0) = -\frac{e}{3}$

Question 10: The subset

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 \leq 1 \text{ and } y \leq x^2\} \subset \mathbb{R}^2$$

☐ is not bounded and not closed

☐ is closed and bounded

☐ is bounded and not closed

☐ is closed and not bounded

Question 11: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = e^{x-y} - x + e^{2y}.$$

Then

- ☐ the function f admits exactly one local minimum in $\mathbb{R}^2$
- ☐ the function f admits exactly one local maximum in $\mathbb{R}^2$
- ☐ the function f admits a stationary point in $\mathbb{R}^2$ that is not a local extremum
- ☐ the function f does not have stationary points in $\mathbb{R}^2$

Question 12: Let D be the subset of $\mathbb{R}^2$ given by

$$D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, y \geq |x|\}.$$

Then the integral

$$\iint_D y \, dx \, dy$$

equals

- ☐ $\frac{7\pi}{6}$
- ☐ $\frac{7\sqrt{2}}{3}$
- ☐ 0
- ☐ $\frac{3\sqrt{2}}{2}$

Question 13: Let $(\mathbf{a}_n)$ be the sequence of elements in $\mathbb{R}^3$ defined by

$$\mathbf{a}_n = \left(\frac{(-1)^n}{n}, (-1)^n, (-1)^n n \right)^T, \quad \text{for all } n \in \{1, 2, 3, \dots\}.$$

Then

- ☐ the sequence has a bounded subsequence
- ☐ the sequence is bounded
- ☐ the sequence has a converging subsequence
- ☐ the limit $\lim_{n \rightarrow \infty} \mathbf{a}_n$ does not exist

Question 14: Let $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{y^2 \sin^2(x)}{x^4 + y^2}.$$

Then

- ☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \frac{1}{2}$
- ☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$
- ☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist
- ☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \frac{\pi^2}{4}$

Question 15: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then

- ☐ f is not continuous at $(0, 0)$
- ☐ f is continuous at $(0, 0)$, but $\frac{\partial f}{\partial x}(0, 0)$ does not exist
- ☐ f is continuous at $(0, 0)$, $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist, but f is not differentiable at $(0, 0)$
- ☐ f is differentiable at $(0, 0)$

Question 16: The solution $y(x)$ of the differential equation

$$y'(x) + \frac{e^x}{e^x + 1} y(x) = \frac{1}{e^x + 1}$$

with initial condition $y(1) = 0$ also satisfies

- ☐ $y(0) = -2$
- ☐ $y(0) = \frac{1}{2}$
- ☐ $y(0) = -\frac{1}{2}$
- ☐ $y(0) = 2$

Question 17: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = xy^3$$

and let $g(x, y) = x^2 + y^2 - 1$. Then, under the constraint $g(x, y) = 0$, the maximal value of f is

- ☐ $\frac{\sqrt{3}}{8}$
- ☐ $\frac{3\sqrt{3}}{16}$
- ☐ $\frac{\sqrt{3}}{16}$
- ☐ $\frac{1}{4}$

Question 18: The solution $y(x)$ of the differential equation

$$y''(x) - 3y'(x) + 2y(x) = 2x^2 - 2x$$

with initial conditions $y(0) = 2$ and $y'(0) = 2$ also satisfies

- ☐ $y(1) = \frac{5}{4} + 2e$
- ☐ $y(1) = 5 - 2e^2$
- ☐ $y(1) = 5$
- ☐ $y(1) = 5 + 2e - 2e^2$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function of class C^3 . Then

$$\frac{\partial^3 f}{\partial x^2 \partial y}(x, y, z) = \frac{\partial^3 f}{\partial x \partial y \partial x}(x, y, z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

☐ TRUE ☐ FALSE

Question 20: Let A and B be two nonempty subsets of $\mathbb{R}^2$. Then the subset C defined by

$$C = \overline{A} \cup \overline{B}$$

is a closed subset of $\mathbb{R}^2$.

☐ TRUE ☐ FALSE

Question 21: Let $f : \mathbb{R} \rightarrow (0, 1)$ be a function such that $\lim_{t \rightarrow 0} f(t) = L > 0$, then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{f(x^2 + y^2)} = 0.$$

☐ TRUE ☐ FALSE

Question 22: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class C^2 and let $\mathbf{a} \in \mathbb{R}^2$ be a stationary point of f . If the two eigenvalues of the Hessian matrix $\text{Hess}_f(\mathbf{a})$, λ_1 and λ_2 , satisfy

$$\lambda_1 \lambda_2 > 0,$$

then $\mathbf{a}$ is a local extremum of f .

☐ TRUE ☐ FALSE

Question 23: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class C^∞ and let p_1 et p_2 be its first and second order Taylor polynomials around $(x_0, y_0) = (0, 0)$. If p_1 is the zero polynomial, then p_2 is also the zero polynomial.

☐ TRUE ☐ FALSE

Question 24: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function and let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $g(x, y) = (f(x, y))^2$. If f is not differentiable at $(x, y) = (0, 0)$, then g is not differentiable at $(x, y) = (0, 0)$.

☐ TRUE ☐ FALSE

Question 25: Let $A = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Then the boundary ∂A is the empty set.

☐ TRUE ☐ FALSE

Question 26: If $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function of class C^2 , then f admits a tangent plane at each point of its graph.

☐ TRUE ☐ FALSE

Question 27: Let D_1 and D_2 be the subsets of $\mathbb{R}^3$ given by

$$D_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4\} \quad \text{and} \quad D_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 9\}.$$

If $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function, then

$$\iiint_{D_1} f(x, y, z) \, dx \, dy \, dz < \iiint_{D_2} f(x, y, z) \, dx \, dy \, dz.$$

☐ TRUE ☐ FALSE

Question 28: Let D be a nonempty, open, and bounded subset of $\mathbb{R}^n$. If $f: D \rightarrow \mathbb{R}$ is a continuous function, then f attains its global maximum on D .

☐ TRUE ☐ FALSE