Duration : 144 minutes

Analysis II Exam

Common part

Spring 2022

Answers

For the **multiple choice** questions, we give

+3 points if your answer is correct,

0 points if you give no answer or more than one,

−1 if your answer is incorrect.

For the **true/false** questions, we give

+1 points if your answer is correct,

0 points if you give no answer or more than one,

−1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: The solution $y(x)$ of the differential equation

$$y''(x) + 2y'(x) - 3y(x) = 9x$$

with initial conditions $y(0) = 4$ and $y'(0) = 3$ also satisfies

☐ $y(1) = 2e^3 - 1$ ☐ $y(1) = 1 - 2e^3$ ■ $y(1) = 6e - 5$ ☐ $y(1) = 5 - 6e$

Question 2: Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(t) = \int_4^{t^2} e^{tx^2} \, dx \, .$$

Then

☐ $F'(2) = 0$ ■ $F'(2) = 4e^{32}$ ☐ $F'(2) = e^{32}$ ☐ $F'(2) = 2e^{32}$

Question 3: Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be the function defined by

$$f(x,y) = \frac{\tan(3x^2 + y^2)}{3x^2 + y^2} \, .$$

Then

■ $\lim_{(x,y)\to(0,0)} f(x,y) = 1$

☐ $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

☐ $\lim_{(x,y)\to(0,0)} f(x,y) = -1$

☐ $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Question 4: Let D be the subset of $\mathbb{R}^2$ given by

$$D = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0 \, , \; y \geq 0 \, , \; x^2 + y^2 \leq 4\right\} .$$

Then the integral

$$\iint_D xy^3 \, dx \, dy$$

equals

■ $\frac{8}{3}$ ☐ $\frac{16}{3}$ ☐ $\frac{16}{5}$ ☐ $\frac{8}{5}$

Question 5: The solution $y(x)$ of the differential equation

$$y'(x) = \big(y(x)\big)^2 \frac{\cos(x)}{\big(2 + \sin(x)\big)^2}$$

that satisfies the initial condition $y\left(\frac{\pi}{2}\right) = \frac{3}{4}$ also satisfies

☐ $y\left(\frac{\pi}{6}\right) = \frac{1}{4}$ ☐ $y\left(\frac{\pi}{6}\right) = \frac{4}{3}$ ■ $y\left(\frac{\pi}{6}\right) = \frac{5}{7}$ ☐ $y\left(\frac{\pi}{6}\right) = -\frac{5}{3}$

Question 6: The subset

$$A = \{(x, 0, z) \in \mathbb{R}^3 : x^2 + 5z^2 < 10\} \subset \mathbb{R}^3$$

- [ ] is bounded and open
- [x] is bounded and not open
- [ ] is not closed and not bounded
- [ ] is closed and open

Question 7: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = 3x^2y - x^3 - 3y^4 .$$

Then

- [ ] $(2, 1)$ is a stationary point for f but is not a local extremum for f
- [ ] $(2, 1)$ is a local minimum for f
- [x] $(2, 1)$ is a local maximum for f
- [ ] $(2, 1)$ is not a stationary point for f

Question 8: Let $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ be functions of n variables and let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ be defined by

$$\boldsymbol{f}(\boldsymbol{x}) = \Big(f_1(\boldsymbol{x}),\, f_2(\boldsymbol{x}), \ldots,\, f_m(\boldsymbol{x})\Big)^{\mathrm{T}} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

Then

- [ ] it is possible for f_j to be non-continuous for some $j \in \{1, \ldots, m\}$ but for f to still be differentiable
- [ ] if the Jacobian matrix of $\boldsymbol{f}$ exists everywhere, then $\boldsymbol{f}$ is differentiable
- [ ] $\boldsymbol{f}$ is always differentiable
- [x] if $\boldsymbol{f}$ is not differentiable, then there exists $j \in \{1, \ldots, m\}$ such that f_j is not a function of class C^1

Question 9 : Let D be the subset of $\mathbb{R}^3$ given by

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 : 0 \le y \le x\,,\ z \ge 0\,,\ 1 \le x^2+y^2+z^2 \le 9\,,\ z^2 \le x^2+y^2 \right\}.$$

The integral

$$\iiint_D \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}\,dx\,dy\,dz$$

equals

- ☐ $\displaystyle\int_1^3 \left(\int_0^{\pi/4} \left(\int_0^{\pi/4} r\sin^2(\theta)\,d\varphi \right) d\theta \right) dr$
- ■ $\displaystyle\int_1^3 \left(\int_{\pi/4}^{\pi/2} \left(\int_0^{\pi/4} r\sin^2(\theta)\,d\varphi \right) d\theta \right) dr$
- ☐ $\displaystyle\int_1^3 \left(\int_0^{\pi/4} \left(\int_{\pi/4}^{\pi/2} r\sin^2(\theta)\,d\varphi \right) d\theta \right) dr$
- ☐ $\displaystyle\int_1^3 \left(\int_0^{\pi/2} \left(\int_0^{\pi/2} r\sin^2(\theta)\,d\varphi \right) d\theta \right) dr$

Question 10 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x\sin\big(|y|\big)}{\sqrt{3x^2+y^2}} & \text{if } (x,y) \ne (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then the directional derivative for f at $(0,0)$ in the direction of the vector $\boldsymbol{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$

- ☐ does not exist
- ☐ is $\dfrac{1}{2}$
- ☐ is 0
- ■ is $\dfrac{1}{2\sqrt{2}}$

Question 11 : The solution $y(x)$ of the differential equation

$$y'(x) - \frac{4}{x}\,y(x) = x^4 + x^2$$

that satisfies the initial condition $y(1) = -2$ also satisfies

☐ $y(2) = 32$ ☐ $y(2) = 8$ ■ $y(2) = -8$ ☐ $y(2) = -16$

Question 12 : Let $(\boldsymbol{a}_n)$ be the sequence of elements in $\mathbb{R}^3$ defined by

$$\boldsymbol{a}_n = \left(2+\frac{1}{n},\ \frac{(-1)^n}{n},\ \frac{3n^2-n+1}{2n^2+e^n+1}\right)^{\mathrm{T}}, \qquad \text{for each } n \in \{1,2,3,\ldots\}.$$

Then

- ☐ the sequence is not bounded
- ☐ the sequence converges to $\left(2, 0, \frac{3}{2}\right)$
- ☐ the sequence is bounded but does not converge
- ■ the sequence converges to $(2, 0, 0)$

Question 13 : The integral

$$\int_0^1 \left(\int_{\sqrt{y}}^1 \cos(x^3)\, dx \right) dy$$

equals

☐ $\frac{1}{3}\left(1-\sin(1)\right)$ ☐ $\sqrt{\sin(1)}$ ■ $\frac{1}{3}\sin(1)$ ☐ $\frac{1}{3}\sqrt{\sin(1)}$

Question 14 : Let $D = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < \frac{1}{4}\right\}$ and let $f : D \to \mathbb{R}$ be the function defined by

$$f(x,y) = \frac{\ln(1+x+y)}{1+x-y}\,.$$

The second order Taylor polynomial for f around $(0,0)$ is

- ■ $p_2(x,y) = x+y-\frac{3}{2}\,x^2+\frac{1}{2}\,y^2-xy$
- ☐ $p_2(x,y) = x+y-\frac{1}{2}\,x^2-\frac{1}{2}\,y^2-xy$
- ☐ $p_2(x,y) = x+y+\frac{3}{2}\,x^2-\frac{3}{2}\,y^2+xy$
- ☐ $p_2(x,y) = x+y+\frac{1}{2}\,x^2-\frac{3}{2}\,y^2-xy$

Question 15 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = xy^2$$

and let $g(x,y) = 2x^2+y^2-6$. Then, under the constraint $g(x,y)=0$,

- ☐ the minimum of f is -8
- ☐ the minimum of f is -2
- ☐ the minimum of f is -6
- ■ the minimum of f is -4

Question 16 : Let $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{g}(x, y, z) = \big(x^2 + y^2 + 1,\ yz\big)^{\mathrm{T}}$$

and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function in $C^1(\mathbb{R}^2)$ such that

$$J_f(u, v) = \begin{pmatrix} \dfrac{2}{u} + 2v & 2u \end{pmatrix}.$$

Then the composition $h = f \circ \boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}$ satisfies:

■ $\dfrac{\partial h}{\partial y}(-1, 0, 1) = 4$

□ $\dfrac{\partial h}{\partial y}(-1, 0, 1) = 0$

□ $\dfrac{\partial h}{\partial y}(-1, 0, 1) = -2$

□ $\dfrac{\partial h}{\partial y}(-1, 0, 1) = 2$

Question 17 : Let $D = \big\{(x, y) \in \mathbb{R}^2 : x > 0\,,\ y > 0\big\}$ and let $f : D \to \mathbb{R}$ be the function defined by

$$f(x, y) = 2x^{\,y} - y^{\,x} - 1\,.$$

The equation $f(x, y) = 0$ implicitly defines a function $y = g(x)$ in a neighborhood of $x = 1$ which satisfies $g(1) = 1$ and $f\big(x, g(x)\big) = 0$. The value of $g(1)$ is then

□ $g'(1) = -2$ ■ $g'(1) = 2$ □ $g'(1) = 0$ □ $g'(1) = -1$

Question 18 : Let S be the surface

$$S = \big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 2xz = 4\big\}$$

and let $z_0 \in \mathbb{R}$ be such that $(1, 1, z_0) \in S$. Then

□ the equation of the tangent plane to S at the point $(1, 1, z_0)$ is $2x + 2y + 2z - 6 = 0$

■ the equation of the tangent plane to S at the point $(1, 1, z_0)$ is $2x + y + z - 4 = 0$

□ the value of z_0 is not unique

□ the equation of the tangent plane to S at the point $(1, 1, z_0)$ is $2z - 4x - 2y + 8 = 0$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subset \mathbb{R}^n$ and let $\boldsymbol{x}_0 \in U$. If $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$, then $\boldsymbol{x}_0$ is a local extremum for f.

□ TRUE ■ FALSE

Question 20 : Let A and B be two nonempty subsets of $\mathbb{R}^n$. If A is a closed set and B is an open set, then the set $C = \{\boldsymbol{x} \in B : \boldsymbol{x} \notin A\}$ is an open set.

■ TRUE □ FALSE

Question 21 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $\boldsymbol{a} \in \mathbb{R}^n$ be such that $f(\boldsymbol{a}) = 1$. Let $(\boldsymbol{a}_k)_{k \geq 1}$ be a sequence in $\mathbb{R}^n$ that converges to $\boldsymbol{a}$. If $\lim_{k\to\infty} f(\boldsymbol{a}_k) = 1$, then f is continuous at $\boldsymbol{a}$.

□ TRUE ■ FALSE

Question 22 : Let A be a nonempty subset of $\mathbb{R}^3$ such that $A \neq \mathbb{R}^3$. If the boundary ∂A is bounded, then A is bounded.

□ TRUE ■ FALSE

Question 23 : If the partial derivatives $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist, then the directional derivative $\nabla_{\boldsymbol{v}} f(0,0)$ exists and is equal to the scalar product of $\nabla f(0,0)$ with $\boldsymbol{v}$, for all $\boldsymbol{v} \in \mathbb{R}^2$ such that $\sqrt{v_1^2 + v_2^2} = 1$.

□ TRUE ■ FALSE

Question 24 : Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subset \mathbb{R}^n$ and let $\boldsymbol{x}_0 \in U$ be such that the partial derivatives $\frac{\partial f}{\partial x_j}$ exist in a neighborhood of $\boldsymbol{x}_0$ for all $j \in \{1, \ldots, n\}$.

If $\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} \frac{\partial f}{\partial x_j}(\boldsymbol{x}) = \frac{\partial f}{\partial x_j}(\boldsymbol{x}_0)$ for all $j \in \{1, \ldots, n\}$, then f is continuous at $\boldsymbol{x}_0$.

■ TRUE □ FALSE

Question 25 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $\boldsymbol{a} \in \mathbb{R}^n$. If the directional derivatives $\nabla_{\boldsymbol{v}} f(\boldsymbol{a})$ exist and are continuous in a neighborhood of $\boldsymbol{a}$ for all $\boldsymbol{v}$, then the function f is differentiable at $\boldsymbol{a}$.

■ TRUE □ FALSE

Question 26: Let $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$. Then

$$\iiint_D \frac{1}{x^2 + y^2 + z^2 + 1}\, dx\, dy\, dz \geq \frac{2\pi}{3}\,.$$

■ TRUE □ FALSE

Question 27: Let $E \subset \mathbb{R}^n$ be a closed subset and let $f : E \to \mathbb{R}$ be a continuous function. If f is bounded and obtains its global maximum on E, then E is bounded.

□ TRUE ■ FALSE

Question 28: Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Let $S = \{\boldsymbol{x} \in \mathbb{R}^3 : F(\boldsymbol{x}) = 0\}$ be the level surface of value 0 of F. If $\boldsymbol{a}, \boldsymbol{b} \in S$ are such that the tangent planes to S at $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\nabla F(\boldsymbol{a}) = \nabla F(\boldsymbol{b})$.

□ TRUE ■ FALSE