Duration : 144 minutes

EPFL

Analysis II Exam

Common part

Spring 2021

Questions

For the **multiple choice** questions, we give

+3 points if your answer is correct,

0 points if you give no answer or more than one,

−1 if your answer is incorrect.

For the **true/false** questions, we give

+1 points if your answer is correct,

0 points if you give no answer or more than one,

−1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Consider the subsets

$$A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1\} \qquad \text{and} \qquad B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 > 1\}.$$

Then the nonempty subset $A \cap B \subset \mathbb{R}^3$ is

- ☐ open
- ☐ bounded
- ☐ closed
- ☐ non bounded

Question 2 : For $f : \mathbb{R}^3 \to \mathbb{R}$ the function defined by

$$f(x, y, z) = xy + 2yz\,.$$

Then the maximum and the minimum of f under the constraint $x^2+y^2+z^2-1 = 0$ are respectively

- ☐ $\frac{6}{\sqrt{20}}$ et $-\frac{6}{\sqrt{20}}$
- ☐ $\frac{\sqrt{2}}{5}$ et $-\frac{\sqrt{2}}{5}$
- ☐ $\frac{\sqrt{5}}{2}$ et $-\frac{\sqrt{5}}{2}$
- ☐ $\frac{5}{\sqrt{20}}$ et $-\frac{3}{\sqrt{20}}$

Question 3 : Let $f : \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{\sin(y^2)}{x^2 + y^2}\,.$$

Then

- ☐ $\lim_{(x,y)\to(0,0)} f(x, y) = 1$
- ☐ $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist
- ☐ $\lim_{(x,y)\to(0,0)} f(x, y) = \frac{1}{2}$
- ☐ $\lim_{(x,y)\to(0,0)} f(x, y) = 0$

Question 4 : The integral

$$\int_0^1 \left(\int_{\arctan(x)}^{\pi/4} \left(\cos^6(y) + \frac{1}{1 + x^2} \right) dy \right) dx$$

equals

☐ $\frac{\pi^2}{32} - \frac{3}{16}$ ☐ $\frac{\pi^2}{32} + \frac{7}{48}$ ☐ $\frac{\pi^2}{16} + \frac{1}{144}$ ☐ $\frac{\pi^2}{16} + \frac{5}{16}$

Question 5: Let $\boldsymbol{g} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{g}(u, v) = \left(v^2,\, u - v\right)^{\mathrm{T}}$$

and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function in C^1 such that

$$J_f(x, y) = \begin{pmatrix} -y\sin(x) & \cos(x) + 2y \end{pmatrix}.$$

Then the composition $h = f \circ \boldsymbol{g} : \mathbb{R}^2 \to \mathbb{R}$ satisfies

- ☐ $\dfrac{\partial h}{\partial v}(1, 0) = 3$
- ☐ $\dfrac{\partial h}{\partial v}(1, 0) = -2$
- ☐ $\dfrac{\partial h}{\partial v}(1, 0) = -3$
- ☐ $\dfrac{\partial h}{\partial v}(1, 0) = 2$

Question 6: The equation of the tangent plane to the surface

$$x^4 + y^4 + z^8 = 3$$

at the point $(1, -1, 1)$ is

- ☐ $x + y + z = 1$
- ☐ $x + y + 2z = 3$
- ☐ $x - y + 2z = 4$
- ☐ $2x - 2y + z = 5$

Question 7: Let $F : (0, +\infty) \to \mathbb{R}$ be the function defined by

$$F(t) = \int_{t^2}^{t} \frac{\sin^2(tx)}{x}\, dx\,.$$

Then

- ☐ $F'(t) = \dfrac{2\sin^2(t^2)}{t} + \dfrac{\sin^2(t^3)}{t}$
- ☐ $F'(t) = \dfrac{\sin^2(t^2)}{t} - \dfrac{2\sin^2(t^3)}{t} + \sin^2(t^2) - \sin^2(t^3)$
- ☐ $F'(t) = \dfrac{2\sin^2(t^2)}{t} - \dfrac{3\sin^2(t^3)}{t}$
- ☐ $F'(t) = \dfrac{\sin^2(t^2)}{t} - \dfrac{\sin^2(t^3)}{t}$

Question 8: Let D be the subset of $\mathbb{R}^3$ given by

$$D = \left\{\, (x, y, z) \in \mathbb{R}^3 : x \geq 0\,,\ y \geq 0\,,\ z \geq 0\,, 1 \leq x^2 + y^2 + z^2 \leq 4\,,\ x^2 + y^2 \leq z^2 \,\right\}.$$

The integral

$$\iiint_D \frac{z^2}{\sqrt{x^2 + y^2 + z^2}}\, dx\, dy\, dz$$

equals

- ☐ $\displaystyle\int_1^2 \left(\int_0^{\pi/4} \left(\int_0^{\pi/2} r^3 \cos^2(\theta) \sin(\theta)\, d\varphi \right) d\theta \right) dr$
- ☐ $\displaystyle\int_1^4 \left(\int_0^{\pi/4} \left(\int_{\pi/4}^{\pi/2} r^3 \cos^2(\theta) \sin(\theta)\, d\varphi \right) d\theta \right) dr$
- ☐ $\displaystyle\int_1^2 \left(\int_0^{\pi/2} \left(\int_{\pi/4}^{\pi/2} r^3 \sin(\theta)\, d\varphi \right) d\theta \right) dr$
- ☐ $\displaystyle\int_1^2 \left(\int_0^{\pi/2} \left(\int_0^{\pi/2} r \cos^2(\theta)\, d\varphi \right) d\theta \right) dr$

Question 9: The solution $y(x)$ of the differential equation

$$\cos(x)y'(x) + \sin(x)y(x) = \cos^2(x)$$

that satisfies the initial condition $y(0) = 2$ also satisfies

- ☐ $y\left(\frac{\pi}{3}\right) = \sqrt{3} + \frac{\pi\sqrt{3}}{6}$
- ☐ $y\left(\frac{\pi}{3}\right) = 1 + \frac{\pi}{6}$
- ☐ $y\left(\frac{\pi}{3}\right) = 1 - \frac{\pi}{6}$
- ☐ $y\left(\frac{\pi}{3}\right) = -\sqrt{3} + \frac{\pi\sqrt{3}}{6}$

Question 10: Let D be the subset of $\mathbb{R}^2$ given by

$$D = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\,,\ x^2 + y^2 \leq 4\right\}.$$

Then the integral

$$\iint_D xy^2(x^2 + y^2)\, dx\, dy$$

equals

- ☐ $\frac{32\sqrt{2}}{21}$
- ☐ $\frac{2^{12}\sqrt{2}}{7}$
- ☐ $\frac{8\sqrt{2}}{9}$
- ☐ $\frac{2^8}{21}$

Question 11: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = \cos(x + y)\cos(x - y)\,.$$

The second order Taylor polynomial for f around $(0, 0)$ is

- ☐ $p_2(x, y) = 1 - x^2 - y^2 - 2xy$
- ☐ $p_2(x, y) = 1 + x^2 + y^2$
- ☐ $p_2(x, y) = 1 - x^2 - y^2$
- ☐ $p_2(x, y) = 1 + x^2 + y^2 - 2xy$

Question 12: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(x, y, z) = zy^5 + 4e^{z-x} - 2\,.$$

The equation $f\big(x, y, z\big) = 0$ defines in a neighbourhood of $(x, y) = (2, -1)$ a function $z = g(x, y)$ which satisfies $g(2, -1) = 2$ and $f\big(x, y, g(x, y)\big) = 0$ as well as

- ☐ $\frac{\partial g}{\partial y}(2, -1) = \frac{3}{10}$
- ☐ $\frac{\partial g}{\partial y}(2, -1) = -\frac{3}{10}$
- ☐ $\frac{\partial g}{\partial y}(2, -1) = \frac{10}{3}$
- ☐ $\frac{\partial g}{\partial y}(2, -1) = -\frac{10}{3}$

Question 13: Let $a \in (0, +\infty)$ be fixed and let $y(x)$ be the solution of the differential equation

$$y'(x) = \left(a + \frac{x}{a}\right) y(x)$$

satisfying the initial condition $y(0) = 1$. Then

- ☐ $y(2) - y(-2) = e^{2a} - e^{-2a}$
- ☐ $y(2) - y(-2) = 0$
- ☐ $y(2)y(-2) = e^{4/a}$
- ☐ $y(2)\, y(-2) = e^{4a}$

Question 14: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \sqrt{|xy|}\,\sin\left(\sqrt{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- ☐ the function f is differentiable at $(0,0)$
- ☐ the function f is continuous at $(0,0)$ but is not continuous on $\mathbb{R}^2$
- ☐ the partial derivatives $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist, but f is not differentiable at $(0,0)$
- ☐ the function f is not continuous at $(0,0)$

Question 15: The solution $u(x)$ of the differential equation

$$u''(x) + 9u(x) = \sin(3x)$$

with initial conditions $u(0) = \frac{\pi}{6}$ and $u'(0) = -\frac{1}{6}$ also satisfies

☐ $u(\pi) = 0$ ☐ $u(\pi) = \frac{\pi - 1}{6}$ ☐ $u(\pi) = \frac{\pi}{6}$ ☐ $u(\pi) = \frac{1}{6}$

Question 16: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^4}{x^4+y^8} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then the directional derivative for f at $(0,0)$ in the direction of the vector $\boldsymbol{v} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^{\mathrm{T}}$

- ☐ is $\frac{9}{2}$
- ☐ is $\frac{3}{4}$
- ☐ does not exist
- ☐ is 0

Question 17: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = x^3 + 2y^3 + 3x^2y + 3xy^2 - x - 2y\,.$$

Then

- ☐ $(-2/\sqrt{3}\,,\ 1/\sqrt{3}\,)$ is a local extremum for f
- ☐ f has exactly two stationary points
- ☐ f does not have a local maximum
- ☐ f admits at least one local maximum and at least one local minimum

Question 18 : Let $\left(\boldsymbol{a}_n\right)_{n\in\mathbb{N}}$ be the sequence of elements in $\mathbb{R}^2$ defined by

$$\boldsymbol{a}_n = \left(\cos\left(\frac{n\pi}{2}\right),\ \sin\left(\frac{n\pi}{4}\right)\right)^{\mathrm{T}}$$

for each $n \in \mathbb{N}$. Then

- ☐ the subsequence $\left(\boldsymbol{a}_{2k+1}\right)_{k\in\mathbb{N}}$ converges
- ☐ the subsequence $\left(\boldsymbol{a}_{4k+1}\right)_{k\in\mathbb{N}}$ converges
- ☐ the subsequence $\left((-1)^k\boldsymbol{a}_{4k+1}\right)_{k\in\mathbb{N}}$ converges
- ☐ the subsequence $\left((-1)^k\boldsymbol{a}_{2k+1}\right)_{k\in\mathbb{N}}$ converges

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $D = \big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\big\}$. Then

$$\iiint_D \frac{z}{x^2 + y^2 + z^2 + 1}\, dx\, dy\, dz \geq \frac{\pi}{8}\,.$$

☐ TRUE ☐ FALSE

Question 20 : Let

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \qquad \text{and} \qquad E = D\backslash\{(0, 0)\}\,.$$

There exists a continuous function $f : E \to \mathbb{R}$ which is unbounded.

☐ TRUE ☐ FALSE

Question 21 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\nabla f(1, 0) = \big(2,\, -1\big)^{\mathrm{T}}$.
If $\nabla_{\boldsymbol{v}} f(1, 0) = 1$ for $\boldsymbol{v} = \left(\frac{1}{\sqrt{2}}\,,\, \frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$, then f is not differentiable at $(1, 0)$.

☐ TRUE ☐ FALSE

Question 22 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $U \subset \mathbb{R}^n$ be nonempty and open. Then the set $f(U)$ is open.

☐ TRUE ☐ FALSE

Question 23 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class C^2 and let $\boldsymbol{a} \in \mathbb{R}^3$ a stationnary point of f. If the three eigenvalues of the hessien matrix $\mathrm{Hess}_f(\boldsymbol{a})$, λ_1, λ_2 and λ_3, satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 \qquad \text{and} \qquad \lambda_1\lambda_2\lambda_3 = -1\,,$$

then $\boldsymbol{a}$ is not a local extremum of f.

☐ TRUE ☐ FALSE

Question 24 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1. If $\nabla f(1, -1) = \big(0,\, 0\big)^{\mathrm{T}}$, then $(1, -1) \in \mathbb{R}^2$ is a local extremum for f.

☐ TRUE ☐ FALSE

Question 25 : Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function at $(x_0, y_0, z_0) \in \mathbb{R}^3$ and let $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function at $t_0 \in \mathbb{R}$ such that $\boldsymbol{g}(t_0) = \big(x_0,\, y_0,\, z_0\big)^{\mathrm{T}}$. If there exists $c \in \mathbb{R}$ such that

$$F\big(\boldsymbol{g}(t)\big) = c \qquad \text{for each } t \in \mathbb{R},$$

then the scalar product of vectors $\nabla F(x_0, y_0, z_0)$ and $\boldsymbol{g}'(t_0)$ is zero.

☐ TRUE ☐ FALSE

Question 26 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function for which the partial derivatives $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist. If $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$, then f is continuous at the point $(0,0)$.

☐ TRUE ☐ FALSE

Question 27 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^3. If $p_2(x,y) = 2 + x^2 + 2y^2$ is the second order Taylor polynomial of f around $(0,0)$, then $\nabla f(0,0) = \left(0,\ 0\right)^{\mathrm{T}}$.

☐ TRUE ☐ FALSE

Question 28 : Let A and B be nonempty subsets of $\mathbb{R}^n$. Then $\partial(A \cup B) = \partial A \cup \partial B$.

☐ TRUE ☐ FALSE