

Analysis II

Exam

Common part

Spring 2020

Answers

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^3$ is

- ☒ not bounded but closed
- ☐ not bounded and neither closed nor open
- ☐ bounded and closed
- ☐ bounded but neither closed nor open

Question 2: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the functions defined by

$$f(x, y) = y^2 + xy + \sqrt{3}x,$$
$$g(x, y) = x^2 + y^2 + xy - \frac{3}{4}.$$

Then, under the constraint $g(x, y) = 0$,

- ☐ the maximum of f is less than 1
- ☒ the minimum of f is less than $-\sqrt{3}$
- ☐ the function f achieves its minimum at exactly two points
- ☐ the function f achieves its maximum at exactly one point

Question 3: Let $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{x^2y + y^4}{x^4 + y^2}.$$

- ☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 1$
- ☒ f does not have a limit at $(0, 0)$
- ☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = y^2$
- ☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$

Question 4: Let

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}.$$

Then, the integral

$$\iint_D \ln(x^2 + y^2) dx dy$$

equals

- ☒ $\pi \ln(4) - \frac{3\pi}{4}$
- ☐ $\pi \ln(2) - \frac{\pi}{4}$
- ☐ $\pi \ln(4) - \pi$
- ☐ $\pi \ln(16) - \pi$

Question 5: Let $f : (0, +\infty) \rightarrow \mathbb{R}$ be the function defined by

$$f(t) = \int_t^{t^2} \frac{1 - e^{-xt}}{x} dx$$

Then, the derivative of f is given by

- ☒ $f'(t) = \frac{-3e^{-t^3} + 2e^{-t^2} + 1}{t}$
- ☐ $f'(t) = \frac{2 - 2e^{-t^3} - 3e^{-t^2}}{t}$
- ☐ $f'(t) = \frac{-e^{-t^3}(t+1) + 2e^{-t^2} - t + 1}{t^2}$
- ☐ $f'(t) = \frac{1 - 3e^{-t^3} - e^{-t^2}}{t}$

Question 6: Let $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the function defined by

$$\mathbf{f}(x, y, z) = (y^2, z^2, x^2)^T.$$

Then, the Jacobian matrix of the composition $\mathbf{g} = \mathbf{f} \circ \mathbf{f}$ at (x, y, z) is

- ☐ $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 4y^2z & 0 \\ 0 & 0 & 4xz^2 \\ 4x^2y & 0 & 0 \end{pmatrix}$
- ☐ $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 4x^3 & 0 \\ 0 & 0 & 4y^3 \\ 4z^3 & 0 & 0 \end{pmatrix}$
- ☒ $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 0 & 4z^3 \\ 4x^3 & 0 & 0 \\ 0 & 4y^3 & 0 \end{pmatrix}$
- ☐ $J_{\mathbf{g}}(x, y, z) = \begin{pmatrix} 0 & 0 & 4yz \\ 4xz & 0 & 0 \\ 0 & 4xy & 0 \end{pmatrix}$

Question 7: Let the surface $S = \{(x, y, z) \in \mathbb{R}^3 : x + y^2 + z^3 - 1 = 0\}$. The equation of the tangent hyperplane for S at the point $(1, -1, -1)$ is

- ☐ $-(y - 2) + (x + 1) + (z - 3) = 0$
- ☐ $-3z - x - 2y + 2 = 0$
- ☒ $x - 1 - 2(y + 1) + 3(z + 1) = 0$
- ☐ $-(x - 1) + (y + 1) + (z + 1) = 0$

Question 8: Let

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq \sqrt{x^2 + y^2} \right\}.$$

Then, the integral

$$\iiint_D \frac{1}{\sqrt{x^2 + y^2 + z^2}} dx dy dz$$

equals

- ☐ $\frac{\pi}{3}$
- ☒ $\pi \left(1 - \frac{1}{\sqrt{2}}\right)$
- ☐ $\frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right)$
- ☐ $2\pi \left(1 - \frac{1}{\sqrt{2}}\right)$

Question 9: The solution $y(x)$ of the differential equation

$$y'(x) - 2y(x) = -x^2$$

satisfying the initial condition $y(0) = \frac{1}{2}$ also satisfies

☐ $y(1) = \frac{1}{4}e^2 + \frac{3}{4}$

☒ $y(1) = \frac{1}{4}e^2 + \frac{5}{4}$

☐ $y(1) = \frac{1}{2}e^2 + \frac{1}{4}$

☐ $y(1) = \frac{1}{2}e^2 - \frac{1}{4}$

Question 10: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function. Then

☒ $\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x, y) dy \right) dx = \int_0^1 \left(\int_{y^2}^{y^{1/3}} f(x, y) dx \right) dy$

☐ $\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x, y) dy \right) dx = \int_0^1 \left(\int_{y^{1/3}}^{y^2} f(x, y) dx \right) dy$

☐ $\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x, y) dy \right) dx = \int_0^{\sqrt{x}} \left(\int_{y^2}^{y^{1/3}} f(x, y) dx \right) dy$

☐ $\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x, y) dy \right) dx = \int_0^1 \left(\int_0^{y^{1/3}} f(x, y) dx \right) dy$

Question 11: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \cos(2x + 3y + y^2).$$

The Taylor polynomial of order two for f around the point $(0, 0)$ is

☐ $p_2(x, y) = 2x + 3y + y^2$

☐ $p_2(x, y) = 1 - (2x + 3y)^2$

☒ $p_2(x, y) = 1 - 2x^2 - 6xy - \frac{9}{2}y^2$

☐ $p_2(x, y) = 1 - \frac{1}{2}y^2$

Question 12: Let, for each $\beta > 1$, the function $f_\beta : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f_\beta(x, y) = \begin{cases} \frac{|x|^\beta y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then, the directional derivative of f_β at $(0, 0)$ for direction $\mathbf{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T$

☐ is equal to $\frac{1}{2}$ if $\beta = 2$

☐ is equal to a positive real number if $\beta > 1$

☐ is equal to $\frac{1}{2^{(\beta+1)/2}}$ if $\beta < 2$

☒ is equal to 0 if $\beta > 2$

Question 13: The solution $y(x)$ of the differential equation

$$y'(x) = e^{x-y(x)}$$

satisfying the initial condition $y(0) = 0$ also satisfies

☐ $y(2) = \ln(2)$

☒ $y(2) = 2$

☐ $y(2) = \ln(2 - e^2)$

☐ $y(2) = \frac{1}{2}$

Question 14: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then

☒ all the partial derivatives of f exist at $(0, 0)$, but f is not differentiable at $(0, 0)$

☐ f is in C^1

☐ f is differentiable at $(0, 0)$, but f is not in C^1

☐ f is differentiable at $(0, 0)$, but one of its partial derivatives is not continuous at $(0, 0)$

Question 15: Let the set $X \subset \mathbb{R}$ be such that for all $\alpha \in X$, all the solutions $y(x)$ of the differential equation

$$y''(x) + (\alpha + 1)y'(x) + \frac{1}{4}y(x) = 0$$

satisfy $\lim_{x \rightarrow \infty} y(x) = 0$. Then

☒ $X = (-1, +\infty)$

☐ $X = (-2, 0)$

☐ $X = [-1, +\infty)$

☐ $X = \mathbb{R}$

Question 16: Let $\tilde{D} \subset \mathbb{R}^2$ be the domain of definition of the change of coordinates $\Phi : \tilde{D} \rightarrow D$, defined by

$$\Phi(x_1, x_2) = \left(x_1 x_2, \frac{x_1}{x_2} \right)^T.$$

Given that $D = (1, 2) \times (1, 2)$, which among the following sets is the only one that can correspond to $\tilde{D}$?

☐ $\tilde{D} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, \quad x_1 x_2 < 2, \quad x_1 < x_2 < 2x_1 \right\}$

☒ $\tilde{D} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \quad \frac{1}{x_1} < x_2 < \frac{2}{x_1}, \quad x_2 < x_1 < 2x_2 \right\}$

☐ $\tilde{D} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \quad x_1 x_2 < 2, \quad 1 < \frac{x_2}{x_1} < 2 \right\}$

☐ $\tilde{D} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \quad x_2 > \frac{1}{x_1}, \quad x_2 < 2x_1 \right\}$

Question 17: Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$ and let $f : D \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = x^3 + \frac{5}{2}x^2 + y^2.$$

Then

- ☐ the minimum of f equals 2
- ☐ the function f has no maximum
- ☐ the function f has a local minimum at $\left(-\frac{5}{3}, 0\right)$
- ☒ the maximum of f equals 18

Question 18: Let $\text{sign} : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$\text{sign}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

Among the following functions, determine which has a point of discontinuity in its domain of definition

- ☐ $\mathbf{f} : (-1, 2) \rightarrow \mathbb{R}^3$ defined by $\mathbf{f}(t) = \left(t, |t|, \text{sign}(t)(e^t - 1)\right)^T$
- ☐ $\mathbf{f} : (0, 5) \rightarrow \mathbb{R}^2$ defined by $\mathbf{f}(t) = \left(t^2, t^2 \cos\left(\frac{1}{t}\right)\right)^T$
- ☒ $\mathbf{f} : [-\pi, \pi] \rightarrow \mathbb{R}^2$ defined by $\mathbf{f}(t) = \left(\text{sign}(t) \cos(t), \text{sign}(t) \sin(t)\right)^T$
- ☐ $\mathbf{f} : [0, 1] \rightarrow \mathbb{R}^3$ defined by $\mathbf{f}(t) = \left(e^{1/(t-2)}, \text{sign}(t) t, \sin(t)\right)^T$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a $C^1(\mathbb{R})$ function, and let the function $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $F(x, y) = y - f(x)$. Then the gradient of F at $(0, f(0))$ is orthogonal to the tangent to the graph of the function f at $(0, f(0))$.

☒ TRUE ☐ FALSE

Question 20: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function so that the partial derivatives exist at $(0, 0)$. Necessarily, f is continuous at $(0, 0)$.

☐ TRUE ☒ FALSE

Question 21: Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be an everywhere differentiable function on $\mathbb{R}^3$. So the partial derivatives of f exist and are continuous at all points of $\mathbb{R}^3$.

☐ TRUE ☒ FALSE

Question 22: Let y_1 be a solution of the differential equation $y' + y = 1$ and let y_0 be a solution of the differential equation $y' + y = 0$. Then, for all constants $C \in \mathbb{R}$, the function $y = y_0 + Cy_1$ is a solution of $y' + y = 1$.

☐ TRUE ☒ FALSE

Question 23: Let $D \subset \mathbb{R}^2$ be a bounded and closed set and $f: D \rightarrow \mathbb{R}$ a $C^2(D)$ function. Necessarily f achieves its minimum in D .

☒ TRUE ☐ FALSE

Question 24: Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 function such that $F(1, 0) = 0$ and $\frac{\partial F}{\partial x}(1, 0) \neq 0$. Then, there exists an open interval $I \subset \mathbb{R}$ such that $0 \in I$, and a $C^1(I)$ function $f: I \rightarrow \mathbb{R}$, so that $f(0) = 1$ and for all $y \in I$, $F(f(y), y) = 0$.

☒ TRUE ☐ FALSE

Question 25: Let $D \subset \mathbb{R}^2$ be the closed unit disc and $f: D \rightarrow \mathbb{R}$ a continuous function. Then the image of f is the interval $[m, M]$, where m is the global minimum of f on D and M is the global maximum of f on D .

☒ TRUE ☐ FALSE

Question 26: Let $D \subset \mathbb{R}^n$ be an open set and $f : D \rightarrow \mathbb{R}$ be a continuous function. It follows that f is unbounded.

☐ TRUE ☒ FALSE

Question 27: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a C^2 function such that $(0,0,0)$ is a stationary point of f and so that the determinant of the Hessian matrix of f at $(0,0,0)$ is strictly positive. Then, f admits a local minimum at $(0,0,0)$.

☐ TRUE ☒ FALSE

Question 28: Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Then, $\iint_D (x + y) \, dx \, dy = 0$.

☒ TRUE ☐ FALSE