Duration : 144 minutes

Analysis II Exam

Common part

Spring 2020

Questions

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^3$ is

- ☐ not bounded but closed
- ☐ not bounded and neither closed nor open
- ☐ bounded and closed
- ☐ bounded but neither closed nor open

Question 2 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be the functions defined by

$$f(x, y) = y^2 + xy + \sqrt{3}\, x\,,$$
$$g(x, y) = x^2 + y^2 + xy - \frac{3}{4}\,.$$

Then, under the constraint $g(x, y) = 0$,

- ☐ the maximum of f is less than 1
- ☐ the minimum of f is less than $-\sqrt{3}$
- ☐ the function f achieves its minimum at exactly two points
- ☐ the function f achieves its maximum at exactly one point

Question 3 : Let $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{x^2 y + y^4}{x^4 + y^2}\,.$$

- ☐ $\lim_{(x,y)\to(0,0)} f(x, y) = 1$
- ☐ f does not have a limit at $(0, 0)$
- ☐ $\lim_{(x,y)\to(0,0)} f(x, y) = y^2$
- ☐ $\lim_{(x,y)\to(0,0)} f(x, y) = 0$

Question 4 : Let

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}\,.$$

Then, the integral

$$\iint_D \ln(x^2 + y^2)\, dx\, dy$$

equals

☐ $\pi \ln(4) - \frac{3\pi}{4}$ ☐ $\pi \ln(2) - \frac{\pi}{4}$ ☐ $\pi \ln(4) - \pi$ ☐ $\pi \ln(16) - \pi$

Question 5: Let $f : (0, +\infty) \to \mathbb{R}$ be the function defined by

$$f(t) = \int_t^{t^2} \frac{1 - e^{-xt}}{x}\, dx$$

Then, the derivative of f is given by

- ☐ $f'(t) = \dfrac{-3e^{-t^3} + 2e^{-t^2} + 1}{t}$
- ☐ $f'(t) = \dfrac{2 - 2e^{-t^3} - 3e^{-t^2}}{t}$
- ☐ $f'(t) = \dfrac{-e^{-t^3}(t+1) + 2e^{-t^2} - t + 1}{t^2}$
- ☐ $f'(t) = \dfrac{1 - 3e^{-t^3} - e^{-t^2}}{t}$

Question 6: Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{f}(x, y, z) = \left(y^2,\, z^2,\, x^2\right)^{\mathrm{T}}.$$

Then, the Jacobian matrix of the composition $\boldsymbol{g} = \boldsymbol{f} \circ \boldsymbol{f}$ at (x, y, z) is

- ☐ $J_{\boldsymbol{g}}(x, y, z) = \begin{pmatrix} 0 & 4y^2z & 0 \\ 0 & 0 & 4xz^2 \\ 4x^2y & 0 & 0 \end{pmatrix}$
- ☐ $J_{\boldsymbol{g}}(x, y, z) = \begin{pmatrix} 0 & 4x^3 & 0 \\ 0 & 0 & 4y^3 \\ 4z^3 & 0 & 0 \end{pmatrix}$
- ☐ $J_{\boldsymbol{g}}(x, y, z) = \begin{pmatrix} 0 & 0 & 4z^3 \\ 4x^3 & 0 & 0 \\ 0 & 4y^3 & 0 \end{pmatrix}$
- ☐ $J_{\boldsymbol{g}}(x, y, z) = \begin{pmatrix} 0 & 0 & 4yz \\ 4xz & 0 & 0 \\ 0 & 4xy & 0 \end{pmatrix}$

Question 7: Let the surface $S = \left\{(x, y, z) \in \mathbb{R}^3 : x + y^2 + z^3 - 1 = 0\right\}$. The equation of the tangent hyperplane for S at the point $(1, -1, -1)$ is

- ☐ $-(y-2) + (x+1) + (z-3) = 0$
- ☐ $-3z - x - 2y + 2 = 0$
- ☐ $x - 1 - 2(y+1) + 3(z+1) = 0$
- ☐ $-(x-1) + (y+1) + (z+1) = 0$

Question 8: Let

$$D = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1,\; z \geq \sqrt{x^2 + y^2}\,\right\}.$$

Then, the integral

$$\iiint_D \frac{1}{\sqrt{x^2 + y^2 + z^2}}\, dx\, dy\, dz$$

equals

- ☐ $\dfrac{\pi}{3}$
- ☐ $\pi\left(1 - \dfrac{1}{\sqrt{2}}\right)$
- ☐ $\dfrac{2\pi}{3}\left(1 - \dfrac{1}{\sqrt{2}}\right)$
- ☐ $2\pi\left(1 - \dfrac{1}{\sqrt{2}}\right)$

Question 9: The solution $y(x)$ of the differential equation

$$y'(x) - 2y(x) = -x^2$$

satisfying the initial condition $y(0) = \frac{1}{2}$ also satisfies

- ☐ $y(1) = \frac{1}{4}\, e^2 + \frac{3}{4}$
- ☐ $y(1) = \frac{1}{4}\, e^2 + \frac{5}{4}$
- ☐ $y(1) = \frac{1}{2}\, e^2 + \frac{1}{4}$
- ☐ $y(1) = \frac{1}{2}\, e^2 - \frac{1}{4}$

Question 10: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then

- ☐ $\displaystyle\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x,y)\, dy \right) dx = \int_0^1 \left(\int_{y^2}^{y^{1/3}} f(x,y)\, dx \right) dy$
- ☐ $\displaystyle\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x,y)\, dy \right) dx = \int_0^1 \left(\int_{y^{1/3}}^{y^2} f(x,y)\, dx \right) dy$
- ☐ $\displaystyle\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x,y)\, dy \right) dx = \int_0^{\sqrt{x}} \left(\int_{y^2}^{y^{1/3}} f(x,y)\, dx \right) dy$
- ☐ $\displaystyle\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x,y)\, dy \right) dx = \int_0^1 \left(\int_0^{y^{1/3}} f(x,y)\, dx \right) dy$

Question 11: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \cos\left(2x + 3y + y^2\right).$$

The Taylor polynomial of order two for f around the point $(0,0)$ is

- ☐ $p_2(x,y) = 2x + 3y + y^2$
- ☐ $p_2(x,y) = 1 - (2x + 3y)^2$
- ☐ $p_2(x,y) = 1 - 2x^2 - 6xy - \frac{9}{2}\, y^2$
- ☐ $p_2(x,y) = 1 - \frac{1}{2}\, y^2$

Question 12: Let, for each $\beta > 1$, the function $f_\beta : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_\beta(x,y) = \begin{cases} \dfrac{|x|^\beta y}{x^2 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then, the directional derivative of f_β at $(0,0)$ for direction $\boldsymbol{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$

- ☐ is equal to $\dfrac{1}{2}$ if $\beta = 2$
- ☐ is equal to a positive real number if $\beta > 1$
- ☐ is equal to $\dfrac{1}{2^{(\beta+1)/2}}$ if $\beta < 2$
- ☐ is equal to 0 if $\beta > 2$

Question 13 : The solution $y(x)$ of the differential equation

$$y'(x) = e^{x-y(x)}$$

satisfying the initial condition $y(0) = 0$ also satisfies

- ☐ $y(2) = \ln(2)$
- ☐ $y(2) = 2$
- ☐ $y(2) = \ln\big(2 - e^2\big)$
- ☐ $y(2) = \frac{1}{2}$

Question 14 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x\,y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- ☐ all the partial derivatives of f exist at $(0,0)$, but f is not differentiable at $(0,0)$
- ☐ f is in C^1
- ☐ f is differentiable at $(0,0)$, but f is not in C^1
- ☐ f is differentiable at $(0,0)$, but one of its partial derivatives is not continuous at $(0,0)$

Question 15 : Let the set $X \subset \mathbb{R}$ be such that for all $\alpha \in X$, all the solutions $y(x)$ of the differential equation

$$y''(x) + (\alpha+1)y'(x) + \tfrac{1}{4}\,y(x) = 0$$

satisfy $\lim\limits_{x\to\infty} y(x) = 0$. Then

- ☐ $X = (-1, +\infty)$
- ☐ $X = (-2, 0)$
- ☐ $X = [-1, +\infty)$
- ☐ $X = \mathbb{R}$

Question 16 : Let $\widetilde{D} \subset \mathbb{R}^2$ be the domain of definition of the change of coordinates $\Phi : \widetilde{D} \to D$, defined by

$$\Phi(x_1, x_2) = \left(x_1\,x_2,\ \frac{x_1}{x_2}\right)^{\mathrm{T}}.$$

Given that $D = (1,2) \times (1,2)$, which among the following sets is the only one that can correspond to $\widetilde{D}$?

- ☐ $\widetilde{D} = \Big\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, \quad x_1x_2 < 2, \quad x_1 < x_2 < 2x_1\Big\}$
- ☐ $\widetilde{D} = \Big\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0, \quad \dfrac{1}{x_1} < x_2 < \dfrac{2}{x_1}, \quad x_2 < x_1 < 2x_2\Big\}$
- ☐ $\widetilde{D} = \Big\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0, \quad x_1x_2 < 2, \quad 1 < \dfrac{x_2}{x_1} < 2\Big\}$
- ☐ $\widetilde{D} = \Big\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0, \quad x_2 > \dfrac{1}{x_1}, \quad x_2 < 2x_1\Big\}$

Question 17 : Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$ and let $f : D \to \mathbb{R}$ be the function defined by

$$f(x, y) = x^3 + \frac{5}{2}\, x^2 + y^2\,.$$

Then

- ☐ the minimum of f equals 2
- ☐ the function f has no maximum
- ☐ the function f has a local minimum at $\left(-\frac{5}{3}\,,\,0\right)$
- ☐ the maximum of f equals 18

Question 18 : Let $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\mathrm{sign}(t) = \begin{cases} 1 & \text{if } t > 0\,, \\ 0 & \text{if } t = 0\,, \\ -1 & \text{if } t < 0\,. \end{cases}$$

Among the following functions, determine which has a point of discontinuity in its domain of definition

- ☐ $\boldsymbol{f} : (-1, 2) \to \mathbb{R}^3$ defined by $\boldsymbol{f}(t) = \Big(t,\, |t|,\, \mathrm{sign}(t)(e^t - 1)\Big)^{\mathrm{T}}$
- ☐ $\boldsymbol{f} : (0, 5) \to \mathbb{R}^2$ defined by $\boldsymbol{f}(t) = \Big(t^2,\, t^2 \cos\Big(\frac{1}{t}\Big)\Big)^{\mathrm{T}}$
- ☐ $\boldsymbol{f} : [-\pi, \pi] \to \mathbb{R}^2$ defined by $\boldsymbol{f}(t) = \Big(\,\mathrm{sign}(t)\cos(t),\, \mathrm{sign}(t)\sin(t)\Big)^{\mathrm{T}}$
- ☐ $\boldsymbol{f} : [0, 1] \to \mathbb{R}^3$ defined by $\boldsymbol{f}(t) = \Big(e^{1/(t-2)},\, \mathrm{sign}(t)\, t,\, \sin(t)\Big)^{\mathrm{T}}$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a $C^1(\mathbb{R})$ function, and let the function $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = y - f(x)$. Then the gradient of F at $\big(0, f(0)\big)$ is orthogonal to the tangent to the graph of the function f at $\big(0, f(0)\big)$.

☐ TRUE ☐ FALSE

Question 20 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function so that the partial derivatives exist at $(0, 0)$. Necessarily, f is continuous at $(0, 0)$.

☐ TRUE ☐ FALSE

Question 21 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be an everywhere differentiable function on $\mathbb{R}^3$. So the partial derivatives of f exist and are continuous at all points of $\mathbb{R}^3$.

☐ TRUE ☐ FALSE

Question 22 : Let y_1 be a solution of the differential equation $y' + y = 1$ and let y_0 be a solution of the differential equation $y' + y = 0$. Then, for all constants $C \in \mathbb{R}$, the function $y = y_0 + Cy_1$ is a solution of $y' + y = 1$.

☐ TRUE ☐ FALSE

Question 23 : Let $D \subset \mathbb{R}^2$ be a bounded and closed set and $f : D \to \mathbb{R}$ a $C^2(D)$ function. Necessarily f achieves its minimum in D.

☐ TRUE ☐ FALSE

Question 24 : Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function such that $F(1, 0) = 0$ and $\frac{\partial F}{\partial x}(1, 0) \neq 0$. Then, there exists an open interval $I \subset \mathbb{R}$ such that $0 \in I$, and a $C^1(I)$ function $f : I \to \mathbb{R}$, so that $f(0) = 1$ and for all $y \in I$, $F\big(f(y), y\big) = 0$.

☐ TRUE ☐ FALSE

Question 25 : Let $D \subset \mathbb{R}^2$ be the closed unit disc and $f : D \to \mathbb{R}$ a continuous function. Then the image of f is the interval $[m, M]$, where m is the global minimum of f on D and M is the global maximum of f on D.

☐ TRUE ☐ FALSE

Question 26 : Let $D \subset \mathbb{R}^n$ be an open set and $f : D \to \mathbb{R}$ be a continuous function. It follows that f is unbounded.

☐ TRUE ☐ FALSE

Question 27 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function such that $(0, 0, 0)$ is a stationary point of f and so that the determinant of the Hessian matrix of f at $(0, 0, 0)$ is strictly positive. Then, f admits a local minimum at $(0, 0, 0)$.

☐ TRUE ☐ FALSE

Question 28 : Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Then, $\iint_D (x + y)\, dx\, dy = 0$.

☐ TRUE ☐ FALSE