

Analysis II

Exam

Common part

Spring 2019

Answers

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2|x|}{x^2 + |x| + y^2}$$

☐ equals 2

☒ does not exist

☐ equals 1

☐ equals 0

Question 2 : Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the functions defined by

$$f(x, y) = 3x + 5y^2 \quad \text{et} \quad g(x, y) = x^2 + 2y^4 + 2xy^2 - 13.$$

Then, under the constraint $g(x, y) = 0$,

☐ the function f attains its maximum at exactly one point

☒ the function f attains its maximum at $(1, \sqrt{2})$

☐ the function f attains its minimum at exactly two points

☐ the minimum of the function f is positive

Question 3 : Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = x \left| x^3 y - \frac{1}{3} \sin(x) \cos(x) \right|.$$

Then

☒ f is differentiable at $(0, 0)$, but is not differentiable at $(\pi, 0)$

☐ $\frac{\partial f}{\partial y}(0, 0)$ exists, but $\frac{\partial f}{\partial x}(0, 0)$ does not exist

☐ f is of class $C^1(\mathbb{R}^2)$

☐ $\nabla f(0, 0)$ exists, but f is not differentiable at $(0, 0)$

Question 4 : For $a > 0$, consider

$$D = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0 \text{ et } x + y + z \leq a\}.$$

Then, for all $a > 0$, the integral

$$\iiint_D x^2 dx dy dz$$

equals

☒ $\frac{a^5}{60}$

☐ $\frac{a^3}{20}$

☐ $\frac{4\pi}{3}a^3$

☐ $\frac{a^5}{20}$

Question 5: Consider

$$D = \{(x, y) \in \mathbb{R}^2 : x > 0\} \quad \text{and} \quad \tilde{D} = \left\{ (r, \varphi) : r > 0, \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}.$$

For $f : D \rightarrow \mathbb{R}$, let $\tilde{f} : \tilde{D} \rightarrow \mathbb{R}$ be the function defined by $\tilde{f}(r, \varphi) = f(r \cos(\varphi), r \sin(\varphi))$.

Then for each function $f : D \rightarrow \mathbb{R}$ of class $C^1(D)$, all $(x, y) \in D$ and all $(r, \varphi) \in \tilde{D}$ such that $x = r \cos(\varphi)$ and $y = r \sin(\varphi)$, one has

☒ $\left(\frac{\partial \tilde{f}}{\partial r}(r, \varphi)\right)^2 + \frac{1}{r^2} \left(\frac{\partial \tilde{f}}{\partial \varphi}(r, \varphi)\right)^2 = \left(\frac{\partial f}{\partial x}(x, y)\right)^2 + \left(\frac{\partial f}{\partial y}(x, y)\right)^2$

☐ $\left(\frac{\partial \tilde{f}}{\partial r}(r, \varphi)\right)^2 + r^2 \left(\frac{\partial \tilde{f}}{\partial \varphi}(r, \varphi)\right)^2 = \left(\frac{\partial f}{\partial x}(x, y)\right)^2 + \left(\frac{\partial f}{\partial y}(x, y)\right)^2$

☐ $\left(\frac{\partial \tilde{f}}{\partial r}(r, \varphi)\right)^2 + \left(\frac{\partial \tilde{f}}{\partial \varphi}(r, \varphi)\right)^2 = \left(\frac{\partial f}{\partial x}(x, y)\right)^2 + \left(\frac{\partial f}{\partial y}(x, y)\right)^2$

☐ $\left(\frac{\partial \tilde{f}}{\partial r}(r, \varphi)\right)^2 + r^2 \sin(\varphi) \left(\frac{\partial \tilde{f}}{\partial \varphi}(r, \varphi)\right)^2 = \left(\frac{\partial f}{\partial x}(x, y)\right)^2 + \left(\frac{\partial f}{\partial y}(x, y)\right)^2$

Question 6: Given $D = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$, let $\mathbf{u} : D \rightarrow \mathbb{R}^2$ be the function defined by

$$\mathbf{u}(x, y, z) = \left(x^2 + 1 + \sin(yz^2), \frac{y}{x}\right)^T$$

Then the Jacobian matrix $J_{\mathbf{u}}(1, 0, 1)$ is

☐ $\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ ☒ $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Question 7: Let $F : (1, +\infty) \rightarrow \mathbb{R}$ be defined by

$$F(t) = \int_{t^{-1/2}}^{t^{3/2}} \frac{\sin(tx^2)}{x} dx.$$

Then $F'(2)$ equals

☐ $\sin(16) - \frac{1}{2} \sin(1)$ ☐ $\frac{\sin(16) - \sin(1)}{4}$

☐ $\frac{\sqrt{2}-1}{4} \sin(16) + \frac{1-2\sqrt{2}}{4} \sin(1)$ ☒ $\sin(16)$

Question 8: Given

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x \leq y^2 \leq 4\},$$

let $f : D \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = e^{y^3+1}.$$

Then the integral $\iint_D f(x, y) dx dy$ equals

☒ $\frac{1}{3}(e^9 - e)$ ☐ $e^8 - 1$ ☐ $\frac{1}{3}(e^9 + e)$ ☐ $e^8 + 1$

Question 9: The solution $y(x)$ of the differential equation

$$(x^2 + 1)y'(x) + y(x) = 1$$

satisfying the initial condition $y(0) = -3$ also satisfies

☐ $y(\tan(1)) = -1 - 3e$

☒ $y(\tan(1)) = 1 - 4e^{-1}$

☐ $y(\tan(1)) = -3e$

☐ $y(\tan(1)) = e - 4e^{-1}$

Question 10: Consider

$$D = \left\{ (r, \varphi) \in \mathbb{R}^2 : 0 \leq \varphi \leq \frac{\pi}{4}, \frac{\sin(\varphi)}{\cos(\varphi)} \leq r \cos(\varphi) \leq 1 \right\} \subset \mathbb{R}^2.$$

Then the integral

$$\iint_D r^2 \sin(\varphi) dr d\varphi$$

is equal to the integral

☐ $\int_0^1 \left(\int_{x^2}^1 y dy \right) dx$

☒ $\int_0^1 \left(\int_0^{x^2} y dy \right) dx$

☐ $\int_0^1 \left(\int_0^{\sqrt{y}} xy dx \right) dy$

☐ $\int_0^1 \left(\int_{\sqrt{y}}^1 \sqrt{x^2 + y^2} y dx \right) dy$

Question 11: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = e^{x^2 y - 1}.$$

The Taylor polynomial of order 2 of f around $(1, 1)$ is

☐ $p_2(x, y) = 2(x - 1) + (y - 1) + 3(x - 1)^2 + (y - 1)^2 + 6(x - 1)(y - 1)$

☐ $p_2(x, y) = 1 + 2x + y + 6x^2 + 4xy + \frac{1}{e} y^2$

☒ $p_2(x, y) = 1 + 2(x - 1) + (y - 1) + 3(x - 1)^2 + 4(x - 1)(y - 1) + \frac{1}{2}(y - 1)^2$

☐ $p_2(x, y) = 1 + 2(x - 1) + (y - 1)$

Question 12: Consider $(x_0, y_0, z_0) = (1, 0, 4)$ and let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y, z) = x^2 + \sin(xy) + z.$$

The equation $f(x, y, z) = 5$ defines in a neighbourhood of $(x_0, z_0) = (1, 4)$ a function $y = g(x, z)$ such that $g(x_0, z_0) = y_0 = 0$ and $f(x, g(x, z), z) = 5$. Furthermore,

☐ $\frac{\partial g}{\partial z}(1, 4) = 1 \quad \text{and} \quad \frac{\partial^2 g}{\partial z^2}(1, 4) = 4$

☐ $\frac{\partial g}{\partial x}(1, 4) = -2 \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2}(1, 4) = 0$

☒ $\frac{\partial g}{\partial x}(1, 4) = -2 \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2}(1, 4) = 2$

☐ $\frac{\partial g}{\partial z}(1, 4) = -2 \quad \text{and} \quad \frac{\partial^2 g}{\partial z^2}(1, 4) = 0$

Question 13: The solution $y(x)$ of the differential equation

$$y'(x) = 4 - (y(x))^2$$

satisfying the initial condition $y(0) = 0$ also satisfies

☐ $y\left(-\frac{1}{4}\right) = 2 \frac{e-1}{e+1}$

☒ $y\left(-\frac{1}{4}\right) = -2 \frac{e-1}{e+1}$

☐ $y\left(-\frac{1}{4}\right) = 2(e^2 - 1)$

☐ $y\left(-\frac{1}{4}\right) = -2 \frac{e+1}{e-1}$

Question 14: Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : -2 \cos(\pi x) + x^2 y + 3e^{xz} + yz = 23\}.$$

The equation of the tangent hyperplane to S at the point $(3, 2, 0)$ is given by

☐ $9x - 12y + z = 36$

☐ $9x + 11y + 12z = 10$

☒ $12x + 9y + 11z = 54$

☐ $12x + 9y + 11z = 18$

Question 15: The solution $u(t)$ of the differential equation

$$u''(t) - 6u'(t) + 9u(t) - 27t = 0$$

satisfying $u(0) = 0$ and $u'(0) = 0$ also satisfies

☐ $u\left(\frac{2}{3}\right) = 0$

☒ $u\left(\frac{2}{3}\right) = 4$

☐ $u\left(\frac{2}{3}\right) = 5e^2 + 4$

☐ $u\left(\frac{2}{3}\right) = e^2$

Question 16: For $\tilde{D} = (0, +\infty) \times (0, 2\pi)$ and $D = \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}$, let $G : \tilde{D} \rightarrow D$ be defined by

$$G(r, \varphi) = (r \cos(\varphi), r \sin(\varphi))^T.$$

Let $f : D \rightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$, be a function of class $C^1(D)$ and let $\tilde{f} : \tilde{D} \rightarrow \mathbb{R}$ be the function defined by $\tilde{f}(r, \varphi) = (f \circ G)(r, \varphi)$. If

$$J_{\tilde{f}}(r, \varphi) = J_{f \circ G}(r, \varphi) = \begin{pmatrix} 2r + \cos(\varphi) \sin(\varphi) & r(\cos^2(\varphi) - \sin^2(\varphi)) \end{pmatrix}$$

for all $(r, \varphi) \in \tilde{D}$, then

☐ $\frac{\partial f}{\partial x}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}$

☐ $\frac{\partial f}{\partial x}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2} - 1$

☒ $\frac{\partial f}{\partial x}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{5\sqrt{2}}{4}$

☐ $\frac{\partial f}{\partial x}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2} + 1$

Question 17: The function

$$f(x, y) = x^3 + y^3 + 3x^2 - 9y^2 - 8$$

☐ attains a local maximum at $(0, 0)$

☐ does not attain either a local maximum or a local minimum at $(0, 6)$

☒ attains a local maximum at $(-2, 0)$

☐ attains a local minimum at $(-2, 6)$

Question 18: The limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y - 3xy^3 + x^5}{x^4 + 2x^2y^2 + y^4}$$

☐ equals 0

☐ equals -3

☐ equals -2

☒ does not exist

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19: Let the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = 4 \sin(x) \cos(x)$. Then the Taylor polynomial of order 2 of f around $(0, 0)$ is the polynomial $p_2(x, y) = 4x$.

☒ TRUE ☐ FALSE

Question 20: Consider $D = [0, 1] \times [-1, 1]$. Then

$$\iint_D \sin(xy) \, dx \, dy = 0.$$

☒ TRUE ☐ FALSE

Question 21: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto f(x, y, z)$, be a function of class C^2 and let $\text{Hess}_f(\mathbf{a})$ be the Hessian matrix of f at $\mathbf{a} \in \mathbb{R}^3$. If $\frac{\partial^2 f}{\partial x^2}(\mathbf{a}) = -2$ and if the determinant of $\text{Hess}_f(\mathbf{a})$ equals -3 , then f admits a local maximum at $\mathbf{a}$.

☐ TRUE ☒ FALSE

Question 22: Let $a \in \mathbb{R}$ be such that $a < -16$. Then the set $\{(x, y) \in \mathbb{R}^2 : 1 \geq -25x^2 - 15y^2 > a\}$ is open.

☒ TRUE ☐ FALSE

Question 23: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class C^1 and consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}.$$

If $f(0, 0) = 3$ and f admits a local minimum at $(0, 0)$, then the equation of the tangent hyperplane to S at the point $(0, 0, 3)$ is given by $z = 3$.

☒ TRUE ☐ FALSE

Question 24: Let D be a closed bounded subset of $\mathbb{R}^2$ and let $f : D \rightarrow \mathbb{R}$ be the function defined by $f(x, y) = \cos(\cos(x - y^2))$. Then f admits a local maximum.

☒ TRUE ☐ FALSE

Question 25: Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and the point $\mathbf{p} \in \mathbb{R}^3$. If the directional derivative of f at $\mathbf{p}$ exists for each vector $\mathbf{v}$, then f is differentiable at $\mathbf{p}$.

☐ TRUE ☒ FALSE

Question 26: Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ be two nonempty subsets. If $E \subset F$, then $\partial E \subset \partial F$.

☐ TRUE ☒ FALSE

Question 27: Given $D = \mathbb{R}^2 \setminus \{(0,0)\}$, let $f : D \rightarrow \mathbb{R}$ be a function so that for all $\theta \in [0, 2\pi]$ we have

$$\lim_{t \rightarrow 0} f(t \cos(\theta), t \sin(\theta)) = 2.$$

Then f can be defined at $(0,0)$ so that it is continuous there.

☐ TRUE ☒ FALSE

Question 28: Let $E \subset \mathbb{R}^2$ be an open and nonempty subset and let $f : E \rightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$, be a function of class $C^2(E)$. Then the function $\frac{\partial f}{\partial x} : E \rightarrow \mathbb{R}$ is differentiable at every point in E .

☒ TRUE ☐ FALSE