Duration : 144 minutes

Analysis II
Exam
Common part
Spring 2019

Questions

For the **multiple choice** questions, we give

+3 points if your answer is correct,

0 points if you give no answer or more than one,

−1 if your answer is incorrect.

For the **true/false** questions, we give

+1 points if your answer is correct,

0 points if you give no answer or more than one,

−1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The limit

$$\lim_{(x,y)\to(0,0)} \frac{2|x|}{x^2+|x|+y^2}$$

☐ equals 2 ☐ does not exist ☐ equals 1 ☐ equals 0

Question 2 : Let $f:\mathbb{R}^2\to\mathbb{R}$ and $g:\mathbb{R}^2\to\mathbb{R}$ be the functions defined by

$$f(x,y)=3x+5y^2 \qquad \text{et} \qquad g(x,y)=x^2+2y^4+2xy^2-13\,.$$

Then, under the constraint $g(x,y)=0$,

☐ the function f attains its maximum at exactly one point

☐ the function f attains its maximum at $\big(1,\sqrt{2}\,\big)$

☐ the function f attains its minimum at exactly two points

☐ the minimum of the function f is positive

Question 3 : Let $f:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$f(x,y)=x\left|x^3y-\tfrac{1}{3}\sin(x)\cos(x)\right|.$$

Then

☐ f is differentiable at $(0,0)$, but is not differentiable at $(\pi,0)$

☐ $\dfrac{\partial f}{\partial y}(0,0)$ exists, but $\dfrac{\partial f}{\partial x}(0,0)$ does not exist

☐ f is of class $C^1(\mathbb{R}^2)$

☐ $\nabla f(0,0)$ exists, but f is not differentiable at $(0,0)$

Question 4 : For $a>0$, consider

$$D=\big\{(x,y,z)\in\mathbb{R}^3 : x\geq 0,\ y\geq 0,\ z\geq 0 \ \text{ et } \ x+y+z\leq a\big\}\,.$$

Then, for all $a>0$, the integral

$$\iiint_D x^2\,dx\,dy\,dz$$

equals

☐ $\dfrac{a^5}{60}$ ☐ $\dfrac{a^3}{20}$ ☐ $\dfrac{4\pi}{3}a^3$ ☐ $\dfrac{a^5}{20}$

Question 5: Consider

$$D = \{(x,y) \in \mathbb{R}^2 : x > 0\} \qquad \text{and} \qquad \widetilde{D} = \left\{(r,\varphi) : r > 0,\ \varphi \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)\right\}.$$

For $f : D \to \mathbb{R}$, let $\widetilde{f} : \widetilde{D} \to \mathbb{R}$ be the function defined by $\widetilde{f}(r,\varphi) = f\big(r\cos(\varphi), r\sin(\varphi)\big)$.
Then for each function $f : D \to \mathbb{R}$ of class $C^1(D)$, all $(x,y) \in D$ and all $(r,\varphi) \in \widetilde{D}$ such that $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$, one has

- ☐ $\left(\dfrac{\partial \widetilde{f}}{\partial r}(r,\varphi)\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial \widetilde{f}}{\partial \varphi}(r,\varphi)\right)^2 = \left(\dfrac{\partial f}{\partial x}(x,y)\right)^2 + \left(\dfrac{\partial f}{\partial y}(x,y)\right)^2$
- ☐ $\left(\dfrac{\partial \widetilde{f}}{\partial r}(r,\varphi)\right)^2 + r^2\left(\dfrac{\partial \widetilde{f}}{\partial \varphi}(r,\varphi)\right)^2 = \left(\dfrac{\partial f}{\partial x}(x,y)\right)^2 + \left(\dfrac{\partial f}{\partial y}(x,y)\right)^2$
- ☐ $\left(\dfrac{\partial \widetilde{f}}{\partial r}(r,\varphi)\right)^2 + \left(\dfrac{\partial \widetilde{f}}{\partial \varphi}(r,\varphi)\right)^2 = \left(\dfrac{\partial f}{\partial x}(x,y)\right)^2 + \left(\dfrac{\partial f}{\partial y}(x,y)\right)^2$
- ☐ $\left(\dfrac{\partial \widetilde{f}}{\partial r}(r,\varphi)\right)^2 + r^2\sin(\varphi)\left(\dfrac{\partial \widetilde{f}}{\partial \varphi}(r,\varphi)\right)^2 = \left(\dfrac{\partial f}{\partial x}(x,y)\right)^2 + \left(\dfrac{\partial f}{\partial y}(x,y)\right)^2$

Question 6: Given $D = \{(x,y,z) \in \mathbb{R}^3 : x \neq 0\}$, let $\boldsymbol{u} : D \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{u}(x,y,z) = \left(x^2 + 1 + \sin\big(yz^2\big),\ \frac{y}{x}\right)^{\mathrm{T}}$$

Then the Jacobian matrix $J_{\boldsymbol{u}}(1,0,1)$ is

☐ $\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Question 7: Let $F : (1,+\infty) \to \mathbb{R}$ be defined by

$$F(t) = \int_{t^{-1/2}}^{t^{3/2}} \frac{\sin(tx^2)}{x}\, dx\,.$$

Then $F'(2)$ equals

- ☐ $\sin(16) - \dfrac{1}{2}\sin(1)$
- ☐ $\dfrac{\sin(16) - \sin(1)}{4}$
- ☐ $\dfrac{\sqrt{2}-1}{4}\sin(16) + \dfrac{1-2\sqrt{2}}{4}\sin(1)$
- ☐ $\sin(16)$

Question 8: Given

$$D = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x \leq y^2 \leq 4\right\},$$

let $f : D \to \mathbb{R}$ be the function defined by

$$f(x,y) = e^{y^3+1}\,.$$

Then the integral $\displaystyle\iint_D f(x,y)\,dx\,dy$ equals

☐ $\frac{1}{3}\big(e^9 - e\big)$ ☐ $e^8 - 1$ ☐ $\frac{1}{3}\big(e^9 + e\big)$ ☐ $e^8 + 1$

Question 9: The solution $y(x)$ of the differential equation

$$(x^2+1)\,y'(x)+y(x)=1$$

satisfying the initial condition $y(0)=-3$ also satisfies

- ☐ $y\big(\tan(1)\big)=-1-3e$
- ☐ $y\big(\tan(1)\big)=1-4e^{-1}$
- ☐ $y\big(\tan(1)\big)=-3e$
- ☐ $y\big(\tan(1)\big)=e-4e^{-1}$

Question 10: Consider

$$D=\left\{(r,\varphi)\in\mathbb{R}^2 : 0\le\varphi\le\frac{\pi}{4},\ \frac{\sin(\varphi)}{\cos(\varphi)}\le r\cos(\varphi)\le 1\right\}\subset\mathbb{R}^2\,.$$

Then the integral

$$\iint_D r^2\sin(\varphi)\,dr\,d\varphi$$

is equal to the integral

- ☐ $\displaystyle\int_0^1\left(\int_{x^2}^1 y\,dy\right)dx$
- ☐ $\displaystyle\int_0^1\left(\int_0^{x^2} y\,dy\right)dx$
- ☐ $\displaystyle\int_0^1\left(\int_0^{\sqrt{y}} xy\,dx\right)dy$
- ☐ $\displaystyle\int_0^1\left(\int_{\sqrt{y}}^1 \sqrt{x^2+y^2}\;y\,dx\right)dy$

Question 11: Let $f:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$f(x,y)=e^{x^2y-1}\,.$$

The Taylor polynomial of order 2 of f around $(1,1)$ is

- ☐ $p_2(x,y)=2(x-1)+(y-1)+3(x-1)^2+(y-1)^2+6(x-1)(y-1)$
- ☐ $p_2(x,y)=1+2x+y+6x^2+4xy+\frac{1}{e}\,y^2$
- ☐ $p_2(x,y)=1+2(x-1)+(y-1)+3(x-1)^2+4(x-1)(y-1)+\frac{1}{2}\,(y-1)^2$
- ☐ $p_2(x,y)=1+2(x-1)+(y-1)$

Question 12: Consider $(x_0,y_0,z_0)=(1,0,4)$ and let $f:\mathbb{R}^3\to\mathbb{R}$ be the function defined by

$$f(x,y,z)=x^2+\sin(xy)+z\,.$$

The equation $f(x,y,z)=5$ defines in a neighbourhood of $(x_0,z_0)=(1,4)$ a function $y=g(x,z)$ such that $g(x_0,z_0)=y_0=0$ and $f\big(x,g(x,z),z\big)=5$. Furthermore,

- ☐ $\dfrac{\partial g}{\partial z}(1,4)=\ \ 1$ and $\dfrac{\partial^2 g}{\partial z^2}(1,4)=4$
- ☐ $\dfrac{\partial g}{\partial x}(1,4)=-2$ and $\dfrac{\partial^2 g}{\partial x^2}(1,4)=0$
- ☐ $\dfrac{\partial g}{\partial x}(1,4)=-2$ and $\dfrac{\partial^2 g}{\partial x^2}(1,4)=2$
- ☐ $\dfrac{\partial g}{\partial z}(1,4)=-2$ and $\dfrac{\partial^2 g}{\partial z^2}(1,4)=0$

Question 13 : The solution $y(x)$ of the differential equation

$$y'(x) = 4 - \big(y(x)\big)^2$$

satisfying the initial condition $y(0) = 0$ also satisfies

- ☐ $y\left(-\frac{1}{4}\right) = 2\,\dfrac{e-1}{e+1}$
- ☐ $y\left(-\frac{1}{4}\right) = -2\,\dfrac{e-1}{e+1}$
- ☐ $y\left(-\frac{1}{4}\right) = 2\,(e^2-1)$
- ☐ $y\left(-\frac{1}{4}\right) = -2\,\dfrac{e+1}{e-1}$

Question 14 : Consider the surface

$$S = \big\{(x,y,z) \in \mathbb{R}^3 : -2\cos(\pi x) + x^2 y + 3e^{xz} + yz = 23\,\big\}.$$

The equation of the tangent hyperplane to S at the point $(3,2,0)$ is given by

- ☐ $9x - 12y + z = 36$
- ☐ $9x + 11y + 12z = 10$
- ☐ $12x + 9y + 11z = 54$
- ☐ $12x + 9y + 11z = 18$

Question 15 : The solution $u(t)$ of the differential equation

$$u''(t) - 6u'(t) + 9u(t) - 27\,t = 0$$

satisfying $u(0) = 0$ and $u'(0) = 0$ also satisfies

- ☐ $u\left(\frac{2}{3}\right) = 0$
- ☐ $u\left(\frac{2}{3}\right) = 4$
- ☐ $u\left(\frac{2}{3}\right) = 5e^2 + 4$
- ☐ $u\left(\frac{2}{3}\right) = e^2$

Question 16 : For $\widetilde{D} = (0,+\infty) \times (0, 2\pi)$ and $D = \mathbb{R}^2 \backslash \big\{(x,0) \in \mathbb{R}^2 : x \geq 0\big\}$, let $G : \widetilde{D} \to D$ be defined by

$$G(r,\varphi) = \big(r\cos(\varphi),\, r\sin(\varphi)\big)^{\mathrm{T}}.$$

Let $f : D \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$, be a function of class $C^1(D)$ and let $\widetilde{f} : \widetilde{D} \to \mathbb{R}$ be the function defined by $\widetilde{f}(r,\varphi) = (f \circ G)(r,\varphi)$. If

$$J_{\widetilde{f}}(r,\varphi) = J_{f\circ G}(r,\varphi) = \begin{pmatrix} 2r + \cos(\varphi)\sin(\varphi) & r\big(\cos^2(\varphi) - \sin^2(\varphi)\big) \end{pmatrix}$$

for all $(r,\varphi) \in \widetilde{D}$, then

- ☐ $\dfrac{\partial f}{\partial x}\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \dfrac{\sqrt{2}}{4}$
- ☐ $\dfrac{\partial f}{\partial x}\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \dfrac{\sqrt{2}}{2} - 1$
- ☐ $\dfrac{\partial f}{\partial x}\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \dfrac{5\sqrt{2}}{4}$
- ☐ $\dfrac{\partial f}{\partial x}\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \dfrac{\sqrt{2}}{2} + 1$

Question 17 : The function

$$f(x,y) = x^3 + y^3 + 3x^2 - 9y^2 - 8$$

- ☐ attains a local maximum at $(0,0)$
- ☐ does not attain either a local maximum or a local minimum at $(0,6)$
- ☐ attains a local maximum at $(-2,0)$
- ☐ attains a local minimum at $(-2,6)$

Question 18 : The limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3y - 3xy^3 + x^5}{x^4 + 2x^2y^2 + y^4}$$

- [ ] equals 0
- [ ] equals -3
- [ ] equals -2
- [ ] does not exist

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 4\sin(x)\cos(x)$. Then the Taylor polynomial of order 2 of f around $(0, 0)$ is the polynomial $p_2(x, y) = 4x$.

☐ TRUE ☐ FALSE

Question 20 : Consider $D = [0, 1] \times [-1, 1]$. Then

$$\iint_D \sin(xy)\, dx\, dy = 0\,.$$

☐ TRUE ☐ FALSE

Question 21 : Let $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto f(x, y, z)$, be a function of class C^2 and let $\mathrm{Hess}_f(\boldsymbol{a})$ be the Hessian matrix of f at $\boldsymbol{a} \in \mathbb{R}^3$. If $\frac{\partial^2 f}{\partial x^2}(\boldsymbol{a}) = -2$ and if the determinant of $\mathrm{Hess}_f(\boldsymbol{a})$ equals -3, then f admits a local maximum at $\boldsymbol{a}$.

☐ TRUE ☐ FALSE

Question 22 : Let $a \in \mathbb{R}$ be such that $a < -16$. Then the set $\{(x, y) \in \mathbb{R}^2 : 1 \geq -25x^2 - 15y^2 > a\}$ is open.

☐ TRUE ☐ FALSE

Question 23 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1 and consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}\,.$$

If $f(0, 0) = 3$ and f admits a local minimum at $(0, 0)$, then the equation of the tangent hyperplane to S at the point $(0, 0, 3)$ is given by $z = 3$.

☐ TRUE ☐ FALSE

Question 24 : Let D be a closed bounded subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be the function defined by $f(x, y) = \cos\big(\cos(x - y^2)\big)$. Then f admits a local maximum.

☐ TRUE ☐ FALSE

Question 25 : Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ and the point $\boldsymbol{p} \in \mathbb{R}^3$. If the directional derivative of f at $\boldsymbol{p}$ exists for each vector $\boldsymbol{v}$, then f is differentiable at $\boldsymbol{p}$.

☐ TRUE ☐ FALSE

Question 26 : Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ be two nonempty subsets. If $E \subset F$, then $\partial E \subset \partial F$.

☐ TRUE ☐ FALSE

Question 27 : Given $D = \mathbb{R}^2 \setminus \{(0,0)\}$, let $f : D \to \mathbb{R}$ be a function so that for all $\theta \in [0, 2\pi]$ we have

$$\lim_{t \to 0} f\big(t\cos(\theta), t\sin(\theta)\big) = 2\,.$$

Then f can be defined at $(0,0)$ so that it is continuous there.

☐ TRUE ☐ FALSE

Question 28 : Let $E \subset \mathbb{R}^2$ be an open and nonempty subset and let $f\colon E \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$, be a function of class $C^2(E)$. Then the function $\frac{\partial f}{\partial x} : E \to \mathbb{R}$ is differentiable at every point in E.

☐ TRUE ☐ FALSE