Duration: 144 minutes

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Analysis II
Exam

Common part

Spring 2018

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: Consider the subsets

$$A = \{(x,y,z) \in \mathbb{R}^3 : x+y+z \le 0\} \qquad \text{and} \qquad B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}.$$

The nonempty subset $A \cap B \subset \mathbb{R}^3$ is

- ☐ open and not bounded
- ☐ closed and not bounded
- ☐ open and bounded
- ■ closed and bounded

Question 2: Let $D = \{(x,y) \in \mathbb{R}^2 : x \ge 0\,,\ |y| \le \min\{1, 2-x\}\}$. Then the integral

$$\iint_D xy^2 \, dx\, dy$$

equals

☐ $\frac{16}{15}$ ☐ $\frac{2}{15}$ ■ $\frac{8}{15}$ ☐ $\frac{4}{15}$

Question 3: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{y}{2+\sin\left(\dfrac{1}{x}\right)} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then

- ☐ the function f is continuous at every point $(x,y) \neq (0,0)$
- ■ the function f is continuous at $(0,0)$
- ☐ $\lim_{x\to 0} f(x,0) \neq \lim_{y\to 0} f(0,y)$
- ☐ $\frac{\partial f}{\partial x}(0,0)$ exists but $\frac{\partial f}{\partial y}(0,0)$ does not exist

Question 4: Let

$$I = \int_0^1 \left(\int_{x^2}^1 \frac{x^{37}}{1+y^{20}} \, dy \right) dx\,.$$

Then

☐ $I = \left(\frac{\pi}{4}\right)^{10}$ ■ $I = \frac{\ln(2)}{760}$ ☐ $I = \frac{\ln(2)}{38}$ ☐ $I = \left(\frac{\pi}{2}\right)^{10}$

Question 5 : Let $\boldsymbol{g} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{g}(u,v) = \big(-v,\, u\big)^{\mathrm{T}}$$

and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function in $C^1(\mathbb{R}^2)$ such that

$$J_f(x,y) = \begin{pmatrix} 3x^2 + y^2 & 2xy + e^{\,y} \end{pmatrix}.$$

The function

$$\begin{aligned} h :\ \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (u,v) &\longmapsto h(u,v) \end{aligned}$$

defined by $h = f \circ \boldsymbol{g}$, satisfies

- ☐ $\dfrac{\partial h}{\partial u}(0,1) = -1$
- ☐ $\dfrac{\partial h}{\partial v}(0,1) = -1$
- ■ $\dfrac{\partial h}{\partial v}(0,1) = -3$
- ☐ $\dfrac{\partial h}{\partial u}(0,1) = -3$

Question 6 : Consider the surface

$$S = \big\{(x,y,z) \in \mathbb{R}^3 : x^4 + y^2 + 4z^2 = 6\big\}.$$

The equation of the tangent hyperplane to S at $(-1,1,1)$ is given by

- ☐ $4x - 2y - 8z - 14 = 0$
- ☐ $x - y - 4z + 6 = 0$
- ■ $2x - y - 4z + 7 = 0$
- ☐ $4x - 2y - 8z + 8 = 0$

Question 7 : Let $(x_0, y_0) = \big(\sqrt{\pi/6}\,,\ \sqrt{2\pi/3}\,\big)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \sin(x^2) + \cos(y^2)\,.$$

The equation $f(x,y) = 0$ defines in a neighbourhood of x_0 a function $y = g(x)$ such that $g(x_0) = y_0$ and $f\big(x, g(x)\big) = 0$. In addition, one has

- ☐ $g'(x_0) = \dfrac{\pi}{2}$
- ☐ $g'(x_0) = -\dfrac{\pi}{2}$
- ■ $g'(x_0) = \dfrac{1}{2}$
- ☐ $g'(x_0) = -\dfrac{1}{2}$

Question 8 : Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(t) = \int_0^{t^2} e^{te^x}\, dx\,.$$

Then one has

- ■ $F'(1) = 3e^e - e$
- ☐ $F'(1) = 2e^e - e$
- ☐ $F'(1) = 4e^e - e$
- ☐ $F'(1) = e^e - e$

Question 9 : The solution $y(x)$ of the differential equation

$$y'(x) + y(x) = x^2 e^{-x}$$

with the initial condition $y(1) = 0$ also satisfies

- ☐ $y(-2) = -\frac{5}{4}\big(e^2 - e^{-4}\big)$
- ☐ $y(-2) = -\frac{1}{4}\big(e^4 - 13e^{-2}\big)$
- ■ $y(-2) = -3e^2$
- ☐ $y(-2) = -3e^{-2}$

Question 10 : Let $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 4\,,\ x \geq 0\,,\ y \geq 0\,,\ z \leq 0\}$.
The integral

$$\iiint_D \frac{xyz}{\sqrt{x^2 + y^2 + z^2}}\, dx\, dy\, dz$$

equals

■ $-\dfrac{31}{40}$ □ 0 □ $-\dfrac{4}{5}$ □ $-\dfrac{31}{30}$

Question 11 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(x, y, z) = x^2 + y^2 + 3z^2 - 5xy\,.$$

Then

□ $(0, 0, 0)$ is a local maximum of f

□ $(0, 0, 0)$ is a local minimum of f

■ $(0, 0, 0)$ is not a local extremum of f

□ $(0, 0, 0)$ is not a stationary point of f

Question 12 : Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{4}\}$ and let $f : D \to \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{1}{1 - x + y + xy}\,.$$

Then the Taylor polynomial of order two of f at the point $(0, 0)$ is:

□ $p_2(x, y) = 1 + x - y + x^2 - 4xy + y^2$

□ $p_2(x, y) = 1 + x - y + x^2 - 2xy + y^2$

□ $p_2(x, y) = 1 + x - y + x^2 - \ xy + y^2$

■ $p_2(x, y) = 1 + x - y + x^2 - 3xy + y^2$

Question 13 : The solution $y(x)$ of the differential equation

$$y'(x) + 6x^5\big(y(x)\big)^2 = 0$$

with the initial condition $y(1) = \frac{1}{3}$ also satisfies

□ $y(\sqrt{2}) = \dfrac{1}{3\,e^7}$ □ $y(\sqrt{2}) = -\dfrac{13}{24}$ □ $y(\sqrt{2}) = -\dfrac{1}{4}$ ■ $y(\sqrt{2}) = \dfrac{1}{10}$

Question 14 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- ☐ f is differentiable at $(0,0)$, but f is not in $C^1(\mathbb{R}^2)$
- ☐ the partial derivatives $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ do not exist
- ■ the partial derivatives $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist, but f is not differentiable at $(0,0)$
- ☐ f is in $C^1(\mathbb{R}^2)$

Question 15 : The solution $u(t)$ of the differential equation

$$u''(t) - 2u'(t) + 5u(t) = 17\sin(2t)$$

with the initial conditions $u(0) = 1$ and $u(0) = 1$ also satisfies

■ $u(\pi) = 4 - 3e^{\pi}$ ☐ $u(\pi) = 4$ ☐ $u(\pi) = -4 + 5e^{\pi}$ ☐ $u(\pi) = 4 - 4e^{\pi}$

Question 16 : Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{f}(x,y,z) = \Big((1+x^2)^y\,,\; y^2 + x \Big)^{\mathrm{T}}.$$

The Jacobian matrix of $\boldsymbol{f}$ evaluated at $(x,y,z) \in \mathbb{R}^3$ is

- ■ $J_{\boldsymbol{f}}(x,y,z) = \begin{pmatrix} (1+x^2)^{y-1} 2xy & (1+x^2)^y \ln(1+x^2) & 0 \\ 1 & 2y & 0 \end{pmatrix}$
- ☐ $J_{\boldsymbol{f}}(x,y,z) = \begin{pmatrix} (1+x^2)^{y-1} 2xy & (1+x^2)^y \ln(1+x^2)\, 2x \\ 1 & 2y \\ 0 & 0 \end{pmatrix}$
- ☐ $J_{\boldsymbol{f}}(x,y,z) = \begin{pmatrix} (1+x^2)^{y-1} 2xy & (1+x^2)^y \ln(1+x^2) \\ 1 & 2y \end{pmatrix}$
- ☐ $J_{\boldsymbol{f}}(x,y,z) = \begin{pmatrix} (1+x^2)^{y-1} 2xy & (1+x^2)^y \ln(1+x^2)\, 2x & 0 \\ 1 & 2y & 0 \end{pmatrix}$

Question 17 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = x^3 y^3$$

and let $g(x,y) = x^4 + 16y^4 - 32$. Then, under the constraint $g(x,y) = 0$,

- ☐ the function f attains its minimum at a single point
- ☐ the function f attains its maximum at $\big(\sqrt[4]{7}\,,\, \sqrt{5}/2\big)$
- ☐ the function f attains its maximum at exactly 4 points
- ■ the function f attains its minimum at $(-2,1)$

Question 18 : The limit $\lim_{(x,y)\to(0,0)} \left(\frac{x^2y}{x^2+y^2} + \frac{xy^3}{x^2+y^6} \right)$

☐ vaut 1 ☐ vaut 2 ■ n'existe pas ☐ vaut 0

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $D \subset \mathbb{R}^n$ be a nonempty subset, closed and bounded and $f : D \to \mathbb{R}$ a continuous function, then

$$-\int_D \big|f(\boldsymbol{x})\big|\,d\boldsymbol{x} \;\leq\; \int_D f(\boldsymbol{x})\,d\boldsymbol{x} \;\leq\; \int_D \big|f(\boldsymbol{x})\big|\,d\boldsymbol{x}$$

■ TRUE □ FALSE

Question 20 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\left(\boldsymbol{a}_k\right)_{k\geq 1}$ a sequence in $\mathbb{R}^n$ such that

$$\lim_{k\to+\infty} \boldsymbol{a}_k = \boldsymbol{a} \in \mathbb{R}^n\,.$$

If $\lim_{k\to+\infty} f(\boldsymbol{a}_k) = 1$, then $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = 1$.

■ TRUE □ FALSE

Question 21 : Let E be a subset of $\mathbb{R}^n$ and let $\overline{E}$ be its closure. If $E = \overline{E}$, then the boundary of E is empty.

□ TRUE ■ FALSE

Question 22 : If y_1 and y_2 are two solutions on a nonempty open interval $I \subset \mathbb{R}$ of the differential equation $y'(x) - \cos\big(y(x)\big) = 0$, then $y_1 + y_2$ is also a solution of this equation on I.

□ TRUE ■ FALSE

Question 23 : If $\boldsymbol{a} \in \mathbb{R}^n$ is a local minimum of the function $f : \mathbb{R}^n \to \mathbb{R}$, then $\boldsymbol{a}$ is a local maximum of the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(\boldsymbol{x}) = -f(\boldsymbol{x})$.

■ TRUE □ FALSE

Question 24 : Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a function and let $\boldsymbol{a} \in \mathbb{R}^n$. If the partial derivatives $\dfrac{\partial f_i}{\partial x_j}(\boldsymbol{a})$ exist for all $i \in \{1,\ldots,m\}$ anf $j \in \{1,\ldots,n\}$, then $\boldsymbol{f}$ is differentiable at $\boldsymbol{a}$.

□ TRUE ■ FALSE

Question 25 : Let $D = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 \leq 1 + y^2 + z^2\right\}$. If $f : D \to \mathbb{R}$ is a continuous function on D, then f is bounded.

□ TRUE ■ FALSE

Question 26 : Let $\left(\boldsymbol{u}_k\right)_{k\geq 1}$ and $\left(\boldsymbol{v}_k\right)_{k\geq 1}$ be two sequences in $\mathbb{R}^n$.
If $\left(\boldsymbol{u}_k + \boldsymbol{v}_k\right)_{k\geq 1}$ is a convergent sequence, then $\left(\boldsymbol{u}_k\right)_{k\geq 1}$ et $\left(\boldsymbol{v}_k\right)_{k\geq 1}$ converge also.

□ TRUE ■ FALSE

Question 27 : If $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, then $\displaystyle\iint_D e^{x^2}\, dx\, dy \leq \pi e$.

■ TRUE ☐ FALSE

Question 28 : Let $(\boldsymbol{x}_k)_{k \geq 1}$ be a sequence in $\mathbb{R}^n$ such that $\|\boldsymbol{x}_k\| = 1$ for all $k \geq 1$. Then there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $\|\boldsymbol{x}\| = 1$ and $\lim_{k \to +\infty} \boldsymbol{x}_k = \boldsymbol{x}$.

☐ TRUE ■ FALSE