Analysis II
Exam
Common part
Spring 2018

Questions

For the **multiple choice** questions, we give

+3 points if your answer is correct,

0 points if you give no answer or more than one,

−1 if your answer is incorrect.

For the **true/false** questions, we give

+1 points if your answer is correct,

0 points if you give no answer or more than one,

−1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: Consider the subsets
$$A = \{(x, y, z) \in \mathbb{R}^3 : x + y + z \leq 0\} \qquad \text{and} \qquad B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$
The nonempty subset $A \cap B \subset \mathbb{R}^3$ is

☐ open and not bounded

☐ closed and not bounded

☐ open and bounded

☐ closed and bounded

Question 2: Let $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ |y| \leq \min\{1, 2 - x\}\,\}$. Then the integral
$$\iint_D xy^2 \, dx \, dy$$
equals

☐ $\dfrac{16}{15}$ ☐ $\dfrac{2}{15}$ ☐ $\dfrac{8}{15}$ ☐ $\dfrac{4}{15}$

Question 3: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by
$$f(x, y) = \begin{cases} \dfrac{y}{2 + \sin\left(\dfrac{1}{x}\right)} & \text{if } x \neq 0\,, \\[4mm] 0 & \text{if } x = 0\,. \end{cases}$$
Then

☐ the function f is continuous at every point $(x, y) \neq (0, 0)$

☐ the function f is continuous at $(0, 0)$

☐ $\lim\limits_{x \to 0} f(x, 0) \neq \lim\limits_{y \to 0} f(0, y)$

☐ $\dfrac{\partial f}{\partial x}(0, 0)$ exists but $\dfrac{\partial f}{\partial y}(0, 0)$ does not exist

Question 4: Let
$$I = \int_0^1 \left(\int_{x^2}^1 \frac{x^{37}}{1 + y^{20}} \, dy \right) dx\,.$$
Then

☐ $I = \left(\dfrac{\pi}{4}\right)^{10}$ ☐ $I = \dfrac{\ln(2)}{760}$ ☐ $I = \dfrac{\ln(2)}{38}$ ☐ $I = \left(\dfrac{\pi}{2}\right)^{10}$

Question 5: Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by
$$g(u, v) = \left(-v,\ u\right)^{\mathrm{T}}$$
and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function in $C^1(\mathbb{R}^2)$ such that
$$J_f(x, y) = \left(\ 3x^2 + y^2 \quad 2xy + e^y\ \right).$$
The function
$$h : \ \mathbb{R}^2 \ \longrightarrow \ \ \mathbb{R}$$
$$(u, v) \longmapsto h(u, v)$$
defined by $h = f \circ g$, satisfies

☐ $\dfrac{\partial h}{\partial u}(0, 1) = -1$ ☐ $\dfrac{\partial h}{\partial v}(0, 1) = -1$ ☐ $\dfrac{\partial h}{\partial v}(0, 1) = -3$ ☐ $\dfrac{\partial h}{\partial u}(0, 1) = -3$

Question 6: Consider the surface
$$S = \left\{(x, y, z) \in \mathbb{R}^3 : x^4 + y^2 + 4z^2 = 6\right\}.$$
The equation of the tangent hyperplane to S at $(-1, 1, 1)$ is given by

☐ $4x - 2y - 8z - 14 = 0$ ☐ $x - y - 4z + 6 = 0$

☐ $2x - y - 4z + 7 = 0$ ☐ $4x - 2y - 8z + 8 = 0$

Question 7: Let $(x_0, y_0) = \left(\sqrt{\pi/6},\ \sqrt{2\pi/3}\ \right)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by
$$f(x, y) = \sin(x^2) + \cos(y^2).$$
The equation $f(x, y) = 0$ defines in a neighbourhood of x_0 a function $y = g(x)$ such that $g(x_0) = y_0$ and $f\left(x, g(x)\right) = 0$. In addition, one has

☐ $g'(x_0) = \dfrac{\pi}{2}$ ☐ $g'(x_0) = -\dfrac{\pi}{2}$ ☐ $g'(x_0) = \dfrac{1}{2}$ ☐ $g'(x_0) = -\dfrac{1}{2}$

Question 8: Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by
$$F(t) = \int_0^{t^2} e^{te^x}\ dx.$$
Then one has

☐ $F'(1) = 3e^e - e$ ☐ $F'(1) = 2e^e - e$ ☐ $F'(1) = 4e^e - e$ ☐ $F'(1) = e^e - e$

Question 9: The solution $y(x)$ of the differential equation
$$y'(x) + y(x) = x^2 e^{-x}$$
with the initial condition $y(1) = 0$ also satisfies

☐ $y(-2) = -\dfrac{5}{4}\left(e^2 - e^{-4}\right)$ ☐ $y(-2) = -\dfrac{1}{4}\left(e^4 - 13e^{-2}\right)$

☐ $y(-2) = -3e^2$ ☐ $y(-2) = -3e^{-2}$

Question 10: Let $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 4, \ x \ge 0, \ y \ge 0, \ z \le 0\}$.

The integral

$$\iiint_D \frac{xyz}{\sqrt{x^2 + y^2 + z^2}} \, dx \, dy \, dz$$

equals

☐ $-\dfrac{31}{40}$ ☐ 0 ☐ $-\dfrac{4}{5}$ ☐ $-\dfrac{31}{30}$

Question 11: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(x, y, z) = x^2 + y^2 + 3z^2 - 5xy .$$

Then

☐ $(0, 0, 0)$ is a local maximum of f

☐ $(0, 0, 0)$ is a local minimum of f

☐ $(0, 0, 0)$ is not a local extremum of f

☐ $(0, 0, 0)$ is not a stationary point of f

Question 12: Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{4}\}$ and let $f : D \to \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{1}{1 - x + y + xy} .$$

Then the Taylor polynomial of order two of f at the point $(0, 0)$ is:

☐ $p_2(x, y) = 1 + x - y + x^2 - 4xy + y^2$

☐ $p_2(x, y) = 1 + x - y + x^2 - 2xy + y^2$

☐ $p_2(x, y) = 1 + x - y + x^2 - xy + y^2$

☐ $p_2(x, y) = 1 + x - y + x^2 - 3xy + y^2$

Question 13: The solution $y(x)$ of the differential equation

$$y'(x) + 6x^5 \big(y(x)\big)^2 = 0$$

with the initial condition $y(1) = \frac{1}{3}$ also satisfies

☐ $y(\sqrt{2}) = \dfrac{1}{3\,e^7}$ ☐ $y(\sqrt{2}) = -\dfrac{13}{24}$ ☐ $y(\sqrt{2}) = -\dfrac{1}{4}$ ☐ $y(\sqrt{2}) = \dfrac{1}{10}$

Question 14: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy}{\sqrt{x^2 + y^2}} & \text{if } (x,y) \neq (0,0), \\[2mm] 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then

☐ f is differentiable at $(0,0)$, but f is not in $C^1(\mathbb{R}^2)$

☐ the partial derivatives $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ do not exist

☐ the partial derivatives $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist, but f is not differentiable at $(0,0)$

☐ f is in $C^1(\mathbb{R}^2)$

Question 15: The solution $u(t)$ of the differential equation

$$u''(t) - 2u'(t) + 5u(t) = 17\sin(2t)$$

with the initial conditions $u(0) = 1$ and $u(0) = 1$ also satisfies

☐ $u(\pi) = 4 - 3e^\pi$ ☐ $u(\pi) = 4$ ☐ $u(\pi) = -4 + 5e^\pi$ ☐ $u(\pi) = 4 - 4e^\pi$

Question 16: Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined by

$$\boldsymbol{f}(x,y,z) = \left((1+x^2)^y,\ y^2 + x \right)^{\mathrm{T}}.$$

The Jacobian matrix of $\boldsymbol{f}$ evaluated at $(x,y,z) \in \mathbb{R}^3$ is

☐ $J_{\boldsymbol{f}}(x,y,z) = \begin{pmatrix} (1+x^2)^{y-1}2xy & (1+x^2)^y \ln(1+x^2) & 0 \\ 1 & 2y & 0 \end{pmatrix}$

☐ $J_{\boldsymbol{f}}(x,y,z) = \begin{pmatrix} (1+x^2)^{y-1}2xy & (1+x^2)^y \ln(1+x^2)\, 2x \\ 1 & 2y \\ 0 & 0 \end{pmatrix}$

☐ $J_{\boldsymbol{f}}(x,y,z) = \begin{pmatrix} (1+x^2)^{y-1}2xy & (1+x^2)^y \ln(1+x^2) \\ 1 & 2y \end{pmatrix}$

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Question 17: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = x^3 y^3$$

and let $g(x,y) = x^4 + 16y^4 - 32$. Then, under the constraint $g(x,y) = 0$,

☐ the function f attains its minimum at a single point

☐ the function f attains its maximum at $(\sqrt[4]{7},\ \sqrt{5}/2)$

☐ the function f attains its maximum at exactly 4 points

☐ the function f attains its minimum at $(-2, 1)$

Question 18 : The limit $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{x^2y}{x^2+y^2}+\frac{xy^3}{x^2+y^6}\right)$

☐ vaut 1   ☐ vaut 2   ☐ n'existe pas   ☐ vaut 0

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $D \subset \mathbb{R}^n$ be a nonempty subset, closed and bounded and $f : D \to \mathbb{R}$ a continuous function, then

$$-\int_D |f(\boldsymbol{x})| \, d\boldsymbol{x} \;\leq\; \int_D f(\boldsymbol{x}) \, d\boldsymbol{x} \;\leq\; \int_D |f(\boldsymbol{x})| \, d\boldsymbol{x}$$

☐ TRUE   ☐ FALSE

Question 20 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $(\boldsymbol{a}_k)_{k \geq 1}$ a sequence in $\mathbb{R}^n$ such that

$$\lim_{k \to +\infty} \boldsymbol{a}_k = \boldsymbol{a} \in \mathbb{R}^n \,.$$

If $\lim_{k \to +\infty} f(\boldsymbol{a}_k) = 1$, then $\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = 1$.

☐ TRUE   ☐ FALSE

Question 21 : Let E be a subset of $\mathbb{R}^n$ and let $\overline{E}$ be its closure. If $E = \overline{E}$, then the boundary of E is empty.

☐ TRUE   ☐ FALSE

Question 22 : If y_1 and y_2 are two solutions on a nonempty open interval $I \subset \mathbb{R}$ of the differential equation $y'(x) - \cos\big(y(x)\big) = 0$, then $y_1 + y_2$ is also a solution of this equation on I.

☐ TRUE   ☐ FALSE

Question 23 : If $\boldsymbol{a} \in \mathbb{R}^n$ is a local minimum of the function $f : \mathbb{R}^n \to \mathbb{R}$, then $\boldsymbol{a}$ is a local maximum of the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(\boldsymbol{x}) = -f(\boldsymbol{x})$.

☐ TRUE   ☐ FALSE

Question 24 : Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a function and let $\boldsymbol{a} \in \mathbb{R}^n$. If the partial derivatives $\dfrac{\partial f_i}{\partial x_j}(\boldsymbol{a})$ exist for all $i \in \{1, \ldots, m\}$ anf $j \in \{1, \ldots, n\}$, then $\boldsymbol{f}$ is differentiable at $\boldsymbol{a}$.

☐ TRUE   ☐ FALSE

Question 25 : Let $D = \big\{ (x, y, z) \in \mathbb{R}^3 : x^2 \leq 1 + y^2 + z^2 \big\}$. If $f : D \to \mathbb{R}$ is a continuous function on D, then f is bounded.

☐ TRUE   ☐ FALSE

Question 26 : Let $(\boldsymbol{u}_k)_{k \geq 1}$ and $(\boldsymbol{v}_k)_{k \geq 1}$ be two sequences in $\mathbb{R}^n$.
If $(\boldsymbol{u}_k + \boldsymbol{v}_k)_{k \geq 1}$ is a convergent sequence, then $(\boldsymbol{u}_k)_{k \geq 1}$ et $(\boldsymbol{v}_k)_{k \geq 1}$ converge also.

☐ TRUE   ☐ FALSE

Question 27: If $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, then $\iint_D e^{x^2} \, dx \, dy \leq \pi e$.

☐ TRUE   ☐ FALSE

Question 28: Let $(x_k)_{k \geq 1}$ be a sequence in $\mathbb{R}^n$ such that $\|x_k\| = 1$ for all $k \geq 1$. Then there exists $x \in \mathbb{R}^n$ such that $\|x\| = 1$ and $\lim_{k \to +\infty} x_k = x$.

☐ TRUE   ☐ FALSE