Duration : 144 minutes

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Analysis II
Exam

Common part

Spring 2017

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The solution $y(x)$ of the differential equation

$$y'(x) + 3x^2 y(x) = -6x^2\, e^{-x^3}$$

for $x \in \mathbb{R}$ with the initial condition $y(1) = 0$ satisfies

■ $y(0) = 2$ ☐ $y(0) = -2$ ☐ $y(0) = e^2 - 1$ ☐ $y(0) = 1 - e^{-2}$

Question 2 : Let $\boldsymbol{\gamma} : [3, 8] \to \mathbb{R}^2$ be the curve defined as

$$\boldsymbol{\gamma}(t) = \left(3t,\, 2t^{3/2}\right)^{\mathrm{T}}.$$

The length of the curve is:

☐ 45 ☐ 540 ■ 38 ☐ 3

Question 3 : Let $f : D \to \mathbb{R}$ be the function defined as

$$f(x, y) = \ln\left(4 - (x + y)^2\right),$$

where $D \subset \mathbb{R}^2$ is the biggest subset of $\mathbb{R}^2$ where $f(x, y)$ is well defined. Then

■ the set D is neither closed nor bounded
☐ the set D is bounded but not closed
☐ the set D is closed and bounded
☐ the set D is closed but not bounded

Question 4 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined as

$$f(x, y, z) = (1 + x)\, y^2\, |z|^3.$$

Then the directional derivative of f at the point $(0, 1, 1)$ in the direction of the vector $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^{\mathrm{T}}$ is equal to

■ $2\sqrt{3}$ ☐ $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^{\mathrm{T}}$ ☐ -4 ☐ 6

Question 5 : Let the function $f : \mathbb{R}^3 \to \mathbb{R}$ and the vector field $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{et} \qquad \boldsymbol{v}(x, y, z) = \left(\tfrac{1}{2}x^2, \tfrac{1}{2}y^2, \tfrac{1}{2}z^2\right)^{\mathrm{T}}.$$

Then, the function

$$g = \operatorname{div}(f\,\boldsymbol{v}) - f\operatorname{div}(\boldsymbol{v})$$

satisfy for all $(x, y, z) \in \mathbb{R}^3$

☐ $g(x, y, z) = f\left(x^2, y^2, z^2\right)$
■ $g(x, y, z) = x^3 + y^3 + z^3$
☐ $g(x, y, z) = 2\langle \nabla f, \boldsymbol{v}\rangle(x, y, z)$, where $\langle\ ,\ \rangle$ is the canonical scalar product on $\mathbb{R}^3$
☐ $g(x, y, z) = f(x, y, z)$

Question 6 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = \begin{cases} \dfrac{x^5 + xy^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

☐ f is not continuous at $(0,0)$

■ $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist but f is not differentiable at $(0,0)$

☐ f is of class $C^1(\mathbb{R}^2)$

☐ f is continuous at $(0,0)$, but $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ do not exist

Question 7 : The solution $u(t)$ of the differential equation

$$u''(t) + 2u'(t) + u(t) = e^{-t}$$

for $t \in \mathbb{R}$ with the initial conditions $u(0) = 1$ and $u'(0) = 0$ satisfies

■ $u(-1) = \frac{1}{2}\,e$

☐ $u(-1) = \frac{5}{2}\,e$

☐ $u(-1) = e^{-1} - e^2 + \frac{1}{2}\,e^{-1}$

☐ $u(-1) = -\frac{1}{2}\,e - e^{-1}$

Question 8 : Let $D = \left\{(x,y) \in \mathbb{R}^2 : 3 \le x^2 + y^2 \le 4\,,\ x \ge 0\,,\ -x \le y \le x\right\}$.
Then the integral

$$\iint_D \left(x^2 - y^2\right) dx\, dy$$

is equal to

■ $\dfrac{7}{4}$

☐ $\dfrac{175}{4}$

☐ $\dfrac{175\pi}{8}$

☐ $\dfrac{7\pi}{8}$

Question 9 : Let D be the region of $\mathbb{R}^3$ bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. Then the integral

$$\iiint_D \sin(x+y+z)\, dx\, dy\, dz$$

is equal to

☐ $\frac{1}{2}\cos(1)$

☐ $\sin(1) + 1$

☐ 1

■ $\frac{1}{2}\cos(1) + \sin(1) - 1$

Question 10 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let

$$I = \int_0^1 \left(\int_{-\sqrt{y}}^0 f(x,y)\,dx \right) dy + \int_1^{\sqrt{2}} \left(\int_{-\sqrt{2-y^2}}^0 f(x,y)\,dx \right) dy\,.$$

Then

- ☐ $I = \int_{-1}^0 \left(\int_{-\sqrt{2-x^2}}^{-x^2} f(x,y)\,dy \right) dx$
- ☐ $I = \int_{-1}^1 \left(\int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy \right) dx$
- ■ $I = \int_{-1}^0 \left(\int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy \right) dx$
- ☐ $I = \int_0^1 \left(\int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy \right) dx$

Question 11 : Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined as

$$\boldsymbol{f}(x,y,z) = \left(x^2 y + e^{yz},\ yz + x\right)^{\mathrm{T}}.$$

The Jacobian matrix of $\boldsymbol{f}$ evaluated at $(x,y,z) \in \mathbb{R}^3$ is

- ☐ $\begin{pmatrix} 2xy & x^2 + ze^{yz} & ye^{yz} \\ 2zy & z & y \end{pmatrix}$
- ■ $\begin{pmatrix} 2xy & x^2 + ze^{yz} & ye^{yz} \\ 1 & z & y \end{pmatrix}$
- ☐ $\begin{pmatrix} 2xy & 2zy \\ x^2 + ze^{yz} & z \\ ye^{yz} & y \end{pmatrix}$
- ☐ $\begin{pmatrix} 2xy & 1 \\ x^2 + ze^{yz} & z \\ ye^{yz} & y \end{pmatrix}$

Question 12 : The solution $u(t)$ of the differential equation

$$u'(t) + \left(u(t)\right)^2 \sin(t) = 0$$

for $t \in \mathbb{R}$ with the initial condition $u(0) = \frac{1}{4}$ satisfies

- ☐ $u(\pi) = \frac{1}{4e^2}$
- ☐ $u(\pi) = \frac{1}{2}$
- ☐ $u(\pi) = \frac{e^2}{4}$
- ■ $u(\pi) = \frac{1}{6}$

Question 13 : The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = x^4 - 4x^3 + 3x^2 + 2xy - y^2$$

has

- ☐ three saddle points
- ☐ one saddle point, one local minimum and one local maximum
- ☐ two saddle points and one local minimum
- ■ two saddle points and one local maximum

Question 14 : Consider $D = \{(u,v) \in \mathbb{R}^2 : u + v \neq 0\}$ and the function $f : D \to \mathbb{R}$ defined as

$$f(u,v) = \ln\big(u^2 + 2uv + v^2\big)\,.$$

Let $\boldsymbol{g} : \mathbb{R}^2 \to D$ be a differentiable function such that $\boldsymbol{g}(0,3) = \big(0,\,-3\big)^{\mathrm{T}}$ and

$$J_{\boldsymbol{g}}(0,3) = \begin{pmatrix} -9 & 0 \\ -3 & 3 \end{pmatrix},$$

where $J_{\boldsymbol{g}}$ is the Jacobian matrix of $\boldsymbol{g}$. If $h = f \circ \boldsymbol{g}$, then

- [ ] $\nabla h(0,3) = \big(6,\,0\big)^{\mathrm{T}}$
- [x] $\nabla h(0,3) = \big(8,\,-2\big)^{\mathrm{T}}$
- [ ] $\nabla h(0,3) = \big(0,\,6\big)^{\mathrm{T}}$
- [ ] $\nabla h(0,3) = \big(0,\,0\big)^{\mathrm{T}}$

Question 15 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = \frac{2x}{x^2 + y^2}$$

and let

$$p_2(x,y) = a_0 + a_1(x-1) + a_2(y+1) + a_3(x-1)^2 + a_4(x-1)(y+1) + a_5(y+1)^2\,,$$

where $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$, be the second order Taylor polynomial of f at the point $(1,-1)$. The coefficient a_4 is equal to

- [ ] $a_4 = 2$
- [x] $a_4 = -1$
- [ ] $a_4 = 1$
- [ ] $a_4 = -2$

Question 16 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined as

$$f(x,y,z) = (x-z)^2 + y^2$$

and let $g(x,y,z) = x^2 + y^2 + z^2 - 4$. Then, under the constraint $g(x,y,z) = 0$,

- [ ] the function f attains its maximum in exactly one point
- [ ] the function f attains its minimum in exactly one point
- [x] the function f attains its maximum in $\big(\sqrt{2}\,,\,0\,,\,-\sqrt{2}\,\big)$
- [ ] the function f is not bounded

Question 17 : Consider the following limits

$$\text{(A)} \quad \lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^8}\,, \qquad \text{(B)} \quad \lim_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^8}\,.$$

Then

- [ ] the two limits do not exist
- [ ] the limit (A) exists but the limit (B) does not exist
- [ ] the two limits exist
- [x] the limit (B) exists but the limit (A) does not exist

Question 18 : The set

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + \frac{1}{2}\, y^2 + \frac{1}{4}\, z^2 = 4 \right\}$$

is the graph of a differentiable function in the neighborhood of the point $\boldsymbol{p} = (-1, 2, -2) \in S$.

The equation of the tangent plane to S at $\boldsymbol{p}$ is:

- ☐ $x + y + z - 3 = 0$
- ☐ $x + y + z + 1 = 0$
- ■ $2x - 2y + z + 8 = 0$
- ☐ $x + y + z - 1 = 0$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $\left(\boldsymbol{u}_n\right)_{n\geq 1}$, with $\boldsymbol{u}_n = (x_n, y_n, z_n) \in \mathbb{R}^3$, be a divergent sequence. Then $\left(x_n\right)_{n\geq 1}$ is a divergent sequence in $\mathbb{R}$.

☐ TRUE ■ FALSE

Question 20 : Let $D = \mathbb{R}^2 \backslash \{(0,0)\}$ and $f : D \to \mathbb{R}$ be a continuous function on D such that $\lim_{t\to 0} f(t,t) = 1$. Then $\lim_{t\to 0} f(t,0) = 1$ and $\lim_{t\to 0} f(0,t) = 1$.

☐ TRUE ■ FALSE

Question 21 : Let $D = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and $f : D \to \mathbb{R}$ be a function. If f is not bounded on D, then f is not continuous on D.

■ TRUE ☐ FALSE

Question 22 : Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ be two differentiable functions. Then for all $\boldsymbol{p} \in \mathbb{R}^3$ the Jacobian matrices of $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{f} + \boldsymbol{g}$ satisfy

$$J_{\boldsymbol{f}+\boldsymbol{g}}(\boldsymbol{p}) = J_{\boldsymbol{f}}(\boldsymbol{p}) + J_{\boldsymbol{g}}(\boldsymbol{p}) .$$

■ TRUE ☐ FALSE

Question 23 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that its directional derivative at the point $\boldsymbol{p} \in \mathbb{R}^n$ exists in the direction of every unitary vector $\boldsymbol{v} \in \mathbb{R}^n$. Then f is differentiable at $\boldsymbol{p}$.

☐ TRUE ■ FALSE

Question 24 : Let $y(x)$ be a solution of the second order linear inhomogeneous differential equation $y''(x) + x^2 y'(x) = 4$. Then for all $C \in \mathbb{R}$, the function $y_1(x) = y(x) + C$ is also a solution of this differential equation.

■ TRUE ☐ FALSE

Question 25 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^2(\mathbb{R}^n)$. Then f is differentiable on $\mathbb{R}^n$.

■ TRUE ☐ FALSE

Question 26 : Let $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable function. Then the composite function $\boldsymbol{f} = \boldsymbol{g} \circ \boldsymbol{g}$ is differentiable, and for all $\boldsymbol{p} \in \mathbb{R}^3$, the Jacobian matrices of $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy

$$\det J_{\boldsymbol{f}}(\boldsymbol{p}) = \left(\det J_{\boldsymbol{g}}(\boldsymbol{p})\right)^2 .$$

☐ TRUE ■ FALSE

Question 27 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx \leq \int_0^2 \left(\int_0^2 f(x,y)\,dy \right) dx\,.$$

☐ TRUE ■ FALSE

Question 28 : Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a function of class $C^2(\mathbb{R}^3)$. Then for every point $\boldsymbol{p} \in \mathbb{R}^3$, the Jacobian matrix $J_{\boldsymbol{f}}(p)$ is a symmetric matrix.

☐ TRUE ■ FALSE