

Duration: 144 minutes

# Analysis II

## Exam

### Common part

### Spring 2016

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## Answers

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For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

## Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

**Question 1:** Consider  $D = \mathbb{R}^2 \setminus \{(0, 0)\}$  and the function  $f : D \rightarrow \mathbb{R}$  defined as

$$f(x, y) = x \cos\left(\frac{1}{x^2 + y^2}\right).$$

Then

☐  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 1$

☐  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = y$

☐  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist

☒  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$

**Question 2:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function defined as

$$f(x, y) = 2x^2 + 2y^2 + 2x + 2y + 1.$$

Then the point  $\mathbf{p} = (-\frac{1}{2}, -\frac{1}{2})$

☐ is not a stationary point of  $f$

☒ is a local minimum of  $f$

☐ is a local maximum of  $f$

☐ is not a local extremum of  $f$

**Question 3:** The solution  $y(x)$  of the differential equation

$$x^2 y'(x) + 4y'(x) - x y(x) + x = 0$$

for  $x \in \mathbb{R}$  with the initial condition  $y(0) = 5$  satisfies

☐  $y(\sqrt{5}) = 1$

☐  $y(\sqrt{5}) = 2$

☒  $y(\sqrt{5}) = 7$

☐  $y(\sqrt{5}) = -7$

**Question 4:** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a function defined as

$$f(x, y, z) = -2xy^3z^4 + 2x^2y^2 - 4$$

and consider the point  $\mathbf{p} = (1, -1, 1)$ . Since  $f(\mathbf{p}) = 0$  and  $\frac{\partial f}{\partial y}(\mathbf{p}) \neq 0$ , the equation  $f(x, y, z) = 0$  defines on a neighborhood of  $(x, z) = (1, 1)$  a function  $y = g(x, z)$  which satisfies  $g(1, 1) = -1$  and  $f(x, g(x, z), z) = 0$ . Moreover

☐  $\frac{\partial g}{\partial z}(1, 1) = 1$

☐  $\frac{\partial g}{\partial z}(1, 1) = -\frac{4}{5}$

☒  $\frac{\partial g}{\partial z}(1, 1) = \frac{4}{5}$

☐  $\frac{\partial g}{\partial z}(1, 1) = 4$

**Question 5:** Let  $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25 \text{ and } y \geq 0\}$  and let  $f : D \rightarrow \mathbb{R}$  be the function defined as

$$f(x, y) = 3y - 4x.$$

Then the global maximum value  $M = \max_{(x,y) \in D} f(x, y)$  of  $f$  on  $D$  and the global minimum value  $m = \min_{(x,y) \in D} f(x, y)$  of  $f$  on  $D$  satisfy

☐  $M = 25$  and  $m = -25$

☐  $M = 20$  and  $m = -25$

☒  $M = 25$  and  $m = -20$

☐  $M = 20$  and  $m = -20$

**Question 6:** Let  $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, y \leq 0, x \leq 0\}$ . Then the integral

$$\iint_D x y^3 dx dy$$

is equal to

☐  $\frac{31}{20}$

☐  $-\frac{21}{8}$

☐  $\frac{3\pi}{4}$

☒  $\frac{21}{8}$

**Question 7:** Let  $\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  the function be defined as

$$\mathbf{h}(u, v) = (u^2, v e^{-u}, e^{-2v})^T$$

and let  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $(x, y, z) \mapsto g(x, y, z)$ , be a function of class  $C^1(\mathbb{R}^3)$ . Then the partial derivative with respect to  $v$  of the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , defined as

$$f(u, v) = g(\mathbf{h}(u, v)),$$

evaluated at the point  $(u, v) = (1, 0)$  is equal to

☒  $\frac{\partial f}{\partial v}(1, 0) = e^{-1} \frac{\partial g}{\partial y}(1, 0, 1) - 2 \frac{\partial g}{\partial z}(1, 0, 1)$

☐  $\frac{\partial f}{\partial v}(1, 0) = \frac{\partial g}{\partial y}(1, 0, 1)$

☐  $\frac{\partial f}{\partial v}(1, 0) = \frac{\partial g}{\partial x}(1, 0, 1) - e^{-1} \frac{\partial g}{\partial y}(1, 0, 1)$

☐  $\frac{\partial f}{\partial v}(1, 0) = e^{-1} \frac{\partial g}{\partial x}(1, 0, 1) - e^{-1} \frac{\partial g}{\partial y}(1, 0, 1)$

**Question 8:** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a function defined as

$$f(x, y, z) = (x^2 + y)z + z^2 - z.$$

Then a vector  $\mathbf{v}$  perpendicular to the level surface of  $f$  through the point  $(2, 0, -1)$  is

☐  $\mathbf{v} = (1, -4, 1)^T$

☒  $\mathbf{v} = (4, 1, -1)^T$

☐  $\mathbf{v} = (-1, -4, 4)^T$

☐  $\mathbf{v} = (-4, 1, 1)^T$

**Question 9:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function defined as

$$f(x, y) = 3y - x^2.$$

The minimum value of  $f$  subject to the constraint  $g(x, y) = 4x^2 + 9y^2 - 36 = 0$  is

☐  $-6$

☐  $-9$

☒  $-10$

☐  $6$

**Question 10:** Let  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the function defined as

$$\mathbf{f}(x, y) = (e^{xy}, \cos(xy), \arctan(x - 2y))^T.$$

Then the Jacobian matrix  $J_{\mathbf{f}}(x, y)$  of  $\mathbf{f}$  evaluated at the point  $\mathbf{p} = (2, 1)$  is

☒  $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & 2e^2 \\ -\sin(2) & -2\sin(2) \\ 1 & -2 \end{pmatrix}$

☐  $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & 2e^2 \\ -\sin(2) & -2\sin(2) \\ 1 & 2 \end{pmatrix}$

☐  $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & -\sin(2) & 1 \\ 2e^2 & -2\sin(2) & 2 \end{pmatrix}$

☐  $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & -\sin(2) & 1 \\ 2e^2 & -2\sin(2) & -2 \end{pmatrix}$

**Question 11:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function and let

$$I = \int_{-1}^1 \left( \int_0^{1-|x|} f(x, y) dy \right) dx.$$

Then

☐  $I = \int_0^1 \left( \int_0^{1-|y|} f(x, y) dx \right) dy$

☐  $I = \int_{-1}^1 \left( \int_0^{1-|y|} f(x, y) dx \right) dy$

☐  $I = \int_{-1}^1 \left( \int_{1-y}^{1+y} f(x, y) dx \right) dy$

☒  $I = \int_0^1 \left( \int_{y-1}^{1-y} f(x, y) dx \right) dy$

**Question 12:** Let  $F : (0, \infty) \rightarrow \mathbb{R}$  be the function defined as

$$F(t) = \int_t^{t^2} \frac{e^{xt}}{x} dx.$$

Then we have

☐  $F'(1) = 3e$

☐  $F'(1) = 1$

☐  $F'(1) = 0$

☒  $F'(1) = e$

**Question 13:** The solution  $u(t)$  of the differential equation

$$u''(t) - u'(t) - 2u(t) = 4t - 2$$

for  $t \in \mathbb{R}$  with the initial conditions  $u(0) = 0$  and  $u'(0) = 3$  satisfies

☒  $u(1) = e^2 - 3e^{-1}$

☐  $u(1) = -2e^{-2} - 2e + 2$

☐  $u(1) = e - e^{-2}$

☐  $u(1) = e^2 - e^{-1}$

**Question 14:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function defined as

$$f(x, y) = x^3 + y^2 - 2x - y$$

and consider the point  $\mathbf{p} = (1, -1)$ . Then the equation of the tangent plane to the graph of  $f$  at  $(\mathbf{p}, f(\mathbf{p}))$  is

☐  $z + 2x + y + 1 = 0$

☒  $z - x + 3y + 3 = 0$

☐  $z + 2x + y - 2 = 0$

☐  $z - x + 3y - 1 = 0$

**Question 15:** Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  be a curve defined as

$$\gamma(t) = (t \cos(t), t \sin(t), t)^T.$$

The length of the arc of  $\gamma$  from the point  $A = (0, 0, 0)$  to the point  $B = (2\pi, 0, 2\pi)$  is equal to

☒  $\int_0^{2\pi} \sqrt{2 + t^2} dt$

☐  $\int_0^\pi \sqrt{2 + t^2} dt$

☐  $\int_0^\pi (2 + t^2) dt$

☐  $\int_0^{2\pi} 2t^2 dt$

**Question 16:** The second order Taylor polynomial of the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as

$$f(x, y) = \sin(xy)$$

at the point  $(0, 1)$  is

☐  $p_2(x, y) = x + y + 2(x - 1)y$

☐  $p_2(x, y) = x - x^2 - (y - 1)^2$

☐  $p_2(x, y) = 1 + x + y + 2(x - 1)y$

☒  $p_2(x, y) = x + x(y - 1)$

**Question 17:** Let  $D = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$  and  $f : D \rightarrow \mathbb{R}$  be the function defined as

$$f(x, y) = \ln(x^2 + y).$$

Then the directional derivative of  $f$  at the point  $(2, 1)$  in the direction of the vector  $\mathbf{e} = \left(\frac{3}{5}, \frac{4}{5}\right)^T$  is equal to

☒  $\frac{16}{25}$

☐  $-\frac{16}{25}$

☐  $\frac{85}{29}$

☐  $-\frac{85}{29}$

**Question 18:** Consider  $D = (0, \infty) \times (0, \infty)$  and the function  $f : D \rightarrow \mathbb{R}$  defined as

$$f(x, y) = |\ln(x) \ln(y)|.$$

Then

☐  $f$  is bounded on  $D$

☐  $f$  is not continuous at  $(1, 1)$

☒ the partial derivatives of  $f$  exist at  $(1, 1)$

☐ the partial derivative of  $f$  with respect to  $y$  does not exist at  $(1, 1)$

## Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

**Question 19:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function of class  $C^2(\mathbb{R}^2)$ . Then, for all  $\mathbf{p} \in \mathbb{R}^2$  we have

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{p}) = \frac{\partial^2 f}{\partial y \partial x}(\mathbf{p}).$$

☒ TRUE ☐ FALSE

**Question 20:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function of class  $C^2(\mathbb{R}^2)$  and let  $\mathbf{p} \in \mathbb{R}^2$ . If  $\mathbf{p}$  is a stationary point of  $f$  and if the determinant of the Hessian matrix  $\text{Hess}_f(\mathbf{p})$  is strictly positive, then  $f$  has a local minimum at  $\mathbf{p}$ .

☐ TRUE ☒ FALSE

**Question 21:** Let  $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ . Then

$$0 \leq \iint_D \frac{x^2}{x^2 + y^4} dx dy \leq \pi.$$

☒ TRUE ☐ FALSE

**Question 22:** The set  $D = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z = 0\}$  is open.

☐ TRUE ☒ FALSE

**Question 23:** Consider a function  $f : D \rightarrow \mathbb{R}$  where  $D \subset \mathbb{R}^n$  is a bounded set. If  $f$  is continuous on  $D$ , then  $f$  has an global maximum on  $D$ .

☐ TRUE ☒ FALSE

**Question 24:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function of class  $C^2(\mathbb{R}^2)$  and let  $\mathbf{p} = (x_0, y_0) \in \mathbb{R}^2$ .

If  $f$  has a local maximum at  $\mathbf{p}$ , then

$$\lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(x_0 + h, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)}{h} = 0.$$

☐ TRUE ☒ FALSE

**Question 25:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function on a closed and bounded set  $D \subset \mathbb{R}^2$  and let  $\tilde{D} \subset \mathbb{R}^2$  be a closed and bounded set. If  $G : \tilde{D} \rightarrow D$  is a bijective function of class  $C^1$  and  $J_G(x, y)$  is the Jacobian matrix of  $G$ , then we have

$$\iint_D f(u, v) du dv = \iint_{\tilde{D}} f(G(x, y)) \left| \det(J_G(x, y)) \right| dx dy,$$

assuming both integrals exist.

☒ TRUE ☐ FALSE

**Question 26:** Consider a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ . If  $f$  is differentiable at every point of  $\mathbb{R}^3$ , then  $f$  is of class  $C^1(\mathbb{R}^3)$ .

☐ TRUE      ☒ FALSE

**Question 27:** Consider a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $f(0,0) = 1$ . If for all fixed  $a \in \mathbb{R}$  we have

$$\lim_{x \rightarrow 0} f(x, ax) = 1,$$

then  $f$  is continuous at  $(0,0)$ .

☐ TRUE      ☒ FALSE

**Question 28:** Let  $p, q : I \rightarrow \mathbb{R}$  be two continuous functions defined on the open interval  $I \subset \mathbb{R}$  and let  $L(u) = u'' + pu' + qu$ . If  $u_h$  is a solution of the differential equation  $L(u) = 0$  on  $I$  and  $u_p$  is a solution of the differential equation  $L(u) = g$  on  $I$ , where  $g(t) = \cos(t^2)$ , then  $3u_p + u_h$  is a solution of the differential equation  $L(u) = 3g$  on  $I$ .

☒ TRUE      ☐ FALSE