

Duration : 144 minutes



Analysis II

Exam

Common part

Spring 2016

Answers

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- 1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- 1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: Consider $D = \mathbb{R}^2 \setminus \{(0, 0)\}$ and the function $f : D \rightarrow \mathbb{R}$ defined as

$$f(x, y) = x \cos\left(\frac{1}{x^2 + y^2}\right).$$

Then

- $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 1$
- $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = y$
- $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist
- $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$

Question 2: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = 2x^2 + 2y^2 + 2x + 2y + 1.$$

Then the point $\mathbf{p} = \left(-\frac{1}{2}, -\frac{1}{2}\right)$

- is not a stationary point of f
- is a local minimum of f
- is a local maximum of f
- is not a local extremum of f

Question 3: The solution $y(x)$ of the differential equation

$$x^2 y'(x) + 4y'(x) - x y(x) + x = 0$$

for $x \in \mathbb{R}$ with the initial condition $y(0) = 5$ satisfies

- $y(\sqrt{5}) = 1$
- $y(\sqrt{5}) = 2$
- $y(\sqrt{5}) = 7$
- $y(\sqrt{5}) = -7$

Question 4: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function defined as

$$f(x, y, z) = -2xy^3z^4 + 2x^2y^2 - 4$$

and consider the point $\mathbf{p} = (1, -1, 1)$. Since $f(\mathbf{p}) = 0$ and $\frac{\partial f}{\partial y}(\mathbf{p}) \neq 0$, the equation $f(x, y, z) = 0$ defines on a neighborhood of $(x, z) = (1, 1)$ a function $y = g(x, z)$ which satisfies $g(1, 1) = -1$ and $f(x, g(x, z), z) = 0$. Moreover

- $\frac{\partial g}{\partial z}(1, 1) = 1$
- $\frac{\partial g}{\partial z}(1, 1) = -\frac{4}{5}$
- $\frac{\partial g}{\partial z}(1, 1) = \frac{4}{5}$
- $\frac{\partial g}{\partial z}(1, 1) = 4$

Question 5: Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25 \text{ and } y \geq 0\}$ and let $f : D \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = 3y - 4x.$$

Then the global maximum value $M = \max_{(x,y) \in D} f(x, y)$ of f on D and the global minimum value $m = \min_{(x,y) \in D} f(x, y)$ of f on D satisfy

- $M = 25$ and $m = -25$
- $M = 20$ and $m = -25$
- $M = 25$ and $m = -20$
- $M = 20$ and $m = -20$

Question 6: Let $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, y \leq 0, x \leq 0\}$. Then the integral

$$\iint_D x y^3 dx dy$$

is equal to

- $\frac{31}{20}$
- $-\frac{21}{8}$
- $\frac{3\pi}{4}$
- $\frac{21}{8}$

Question 7: Let $\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ the function be defined as

$$\mathbf{h}(u, v) = (u^2, v e^{-u}, e^{-2v})^T$$

and let $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto g(x, y, z)$, be a function of class $C^1(\mathbb{R}^3)$. Then the partial derivative with respect to v of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined as

$$f(u, v) = g(\mathbf{h}(u, v)),$$

evaluated at the point $(u, v) = (1, 0)$ is equal to

- $\frac{\partial f}{\partial v}(1, 0) = e^{-1} \frac{\partial g}{\partial y}(1, 0, 1) - 2 \frac{\partial g}{\partial z}(1, 0, 1)$
- $\frac{\partial f}{\partial v}(1, 0) = \frac{\partial g}{\partial y}(1, 0, 1)$
- $\frac{\partial f}{\partial v}(1, 0) = \frac{\partial g}{\partial x}(1, 0, 1) - e^{-1} \frac{\partial g}{\partial y}(1, 0, 1)$
- $\frac{\partial f}{\partial v}(1, 0) = e^{-1} \frac{\partial g}{\partial x}(1, 0, 1) - e^{-1} \frac{\partial g}{\partial y}(1, 0, 1)$

Question 8: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function defined as

$$f(x, y, z) = (x^2 + y)z + z^2 - z.$$

Then a vector \mathbf{v} perpendicular to the level surface of f through the point $(2, 0, -1)$ is

- $\mathbf{v} = (1, -4, 1)^T$
- $\mathbf{v} = (-1, -4, 4)^T$
- $\mathbf{v} = (4, 1, -1)^T$
- $\mathbf{v} = (-4, 1, 1)^T$

Question 9: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function defined as

$$f(x, y) = 3y - x^2.$$

The minimum value of f subject to the constraint $g(x, y) = 4x^2 + 9y^2 - 36 = 0$ is

-6

-9

-10

6

Question 10: Let $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the function defined as

$$\mathbf{f}(x, y) = (e^{xy}, \cos(xy), \arctan(x - 2y))^T.$$

Then the Jacobian matrix $J_{\mathbf{f}}(x, y)$ of \mathbf{f} evaluated at the point $\mathbf{p} = (2, 1)$ is

$J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & 2e^2 \\ -\sin(2) & -2\sin(2) \\ 1 & -2 \end{pmatrix}$

$J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & -\sin(2) & 1 \\ 2e^2 & -2\sin(2) & 2 \end{pmatrix}$

$J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & 2e^2 \\ -\sin(2) & -2\sin(2) \\ 1 & 2 \end{pmatrix}$

$J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} e^2 & -\sin(2) & 1 \\ 2e^2 & -2\sin(2) & -2 \end{pmatrix}$

Question 11: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function and let

$$I = \int_{-1}^1 \left(\int_0^{1-|x|} f(x, y) dy \right) dx.$$

Then

$I = \int_0^1 \left(\int_0^{1-|y|} f(x, y) dx \right) dy$

$I = \int_{-1}^1 \left(\int_0^{1-|y|} f(x, y) dx \right) dy$

$I = \int_{-1}^1 \left(\int_{1-y}^{1+y} f(x, y) dx \right) dy$

$I = \int_0^1 \left(\int_{y-1}^{1-y} f(x, y) dx \right) dy$

Question 12: Let $F : (0, \infty) \rightarrow \mathbb{R}$ be the function defined as

$$F(t) = \int_t^{t^2} \frac{e^{xt}}{x} dx.$$

Then we have

$F'(1) = 3e$

$F'(1) = 1$

$F'(1) = 0$

$F'(1) = e$

Question 13: The solution $u(t)$ of the differential equation

$$u''(t) - u'(t) - 2u(t) = 4t - 2$$

for $t \in \mathbb{R}$ with the initial conditions $u(0) = 0$ and $u'(0) = 3$ satisfies

$u(1) = e^2 - 3e^{-1}$

$u(1) = e - e^{-2}$

$u(1) = -2e^{-2} - 2e + 2$

$u(1) = e^2 - e^{-1}$

Question 14: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = x^3 + y^2 - 2x - y$$

and consider the point $\mathbf{p} = (1, -1)$. Then the equation of the tangent plane to the graph of f at $(\mathbf{p}, f(\mathbf{p}))$ is

$z + 2x + y + 1 = 0$

$z - x + 3y + 3 = 0$

$z + 2x + y - 2 = 0$

$z - x + 3y - 1 = 0$

Question 15: Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ be a curve defined as

$$\gamma(t) = (t \cos(t), t \sin(t), t)^T.$$

The length of the arc of γ from the point $A = (0, 0, 0)$ to the point $B = (2\pi, 0, 2\pi)$ is equal to

$\int_0^{2\pi} \sqrt{2+t^2} dt$

$\int_0^\pi \sqrt{2+t^2} dt$

$\int_0^\pi (2+t^2) dt$

$\int_0^{2\pi} 2t^2 dt$

Question 16: The second order Taylor polynomial of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as

$$f(x, y) = \sin(xy)$$

at the point $(0, 1)$ is

$p_2(x, y) = x + y + 2(x - 1)y$

$p_2(x, y) = x - x^2 - (y - 1)^2$

$p_2(x, y) = 1 + x + y + 2(x - 1)y$

$p_2(x, y) = x + x(y - 1)$

Question 17: Let $D = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$ and $f : D \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = \ln(x^2 + y).$$

Then the directional derivative of f at the point $(2, 1)$ in the direction of the vector $\mathbf{e} = \left(\frac{3}{5}, \frac{4}{5}\right)^T$ is equal to

$\frac{16}{25}$

$-\frac{16}{25}$

$\frac{85}{29}$

$-\frac{85}{29}$

Question 18: Consider $D = (0, \infty) \times (0, \infty)$ and the function $f : D \rightarrow \mathbb{R}$ defined as

$$f(x, y) = |\ln(x) \ln(y)|.$$

Then

f is bounded on D

f is not continuous at $(1, 1)$

the partial derivatives of f exist at $(1, 1)$

the partial derivative of f with respect to y does not exist at $(1, 1)$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$. Then, for all $\mathbf{p} \in \mathbb{R}^2$ we have

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{p}) = \frac{\partial^2 f}{\partial y \partial x}(\mathbf{p}).$$

TRUE FALSE

Question 20: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$ and let $\mathbf{p} \in \mathbb{R}^2$. If \mathbf{p} is a stationary point of f and if the determinant of the Hessian matrix $\text{Hess}_f(\mathbf{p})$ is strictly positive, then f has a local minimum at \mathbf{p} .

TRUE FALSE

Question 21: Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Then

$$0 \leq \iint_D \frac{x^2}{x^2 + y^4} dx dy \leq \pi.$$

TRUE FALSE

Question 22: The set $D = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z = 0\}$ is open.

TRUE FALSE

Question 23: Consider a function $f : D \rightarrow \mathbb{R}$ where $D \subset \mathbb{R}^n$ is a bounded set. If f is continuous on D , then f has an global maximum on D .

TRUE FALSE

Question 24: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$ and let $\mathbf{p} = (x_0, y_0) \in \mathbb{R}^2$.

If f has a local maximum at \mathbf{p} , then

$$\lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(x_0 + h, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)}{h} = 0.$$

TRUE FALSE

Question 25: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function on a closed and bounded set $D \subset \mathbb{R}^2$ and let $\tilde{D} \subset \mathbb{R}^2$ be a closed and bounded set. If $G : \tilde{D} \rightarrow D$ is a bijective function of class C^1 and $J_G(x, y)$ is the Jacobian matrix of G , then we have

$$\iint_D f(u, v) du dv = \iint_{\tilde{D}} f(G(x, y)) \left| \det(J_G(x, y)) \right| dx dy,$$

assuming both integrals exist.

TRUE FALSE

Question 26: Consider a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. If f is differentiable at every point of \mathbb{R}^3 , then f is of class $C^1(\mathbb{R}^3)$.

TRUE FALSE

Question 27: Consider a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f(0,0) = 1$. If for all fixed $a \in \mathbb{R}$ we have

$$\lim_{x \rightarrow 0} f(x, ax) = 1,$$

then f is continuous at $(0,0)$.

TRUE FALSE

Question 28: Let $p, q : I \rightarrow \mathbb{R}$ be two continuous functions defined on the open interval $I \subset \mathbb{R}$ and let $L(u) = u'' + pu' + qu$. If u_h is a solution of the differential equation $L(u) = 0$ on I and u_p is a solution of the differential equation $L(u) = g$ on I , where $g(t) = \cos(t^2)$, then $3u_p + u_h$ is a solution of the differential equation $L(u) = 3g$ on I .

TRUE FALSE