Duration : 144 minutes

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

# Analysis II
# Exam
## Common part
## Spring 2016

---

## Questions

---

For the **multiple choice** questions, we give

- $+3$ points if your answer is correct,
- $0$ points if you give no answer or more than one,
- $-1$ if your answer is incorrect.

For the **true/false** questions, we give

- $+1$ points if your answer is correct,
- $0$ points if you give no answer or more than one,
- $-1$ points if your answer is incorrect.

## Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

**Question 1:** Consider $D = \mathbb{R}^2 \backslash \{(0,0)\}$ and the function $f : D \to \mathbb{R}$ defined as

$$f(x,y) = x \cos\left(\frac{1}{x^2+y^2}\right).$$

Then

- ☐ $\lim_{(x,y)\to(0,0)} f(x,y) = 1$
- ☐ $\lim_{(x,y)\to(0,0)} f(x,y) = y$
- ☐ $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist
- ☐ $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

**Question 2:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = 2x^2 + 2y^2 + 2x + 2y + 1\,.$$

Then the point $\boldsymbol{p} = \left(-\frac{1}{2}, -\frac{1}{2}\right)$

- ☐ is not a stationary point of $f$
- ☐ is a local minimum of $f$
- ☐ is a local maximum of $f$
- ☐ is not a local extremum of $f$

**Question 3:** The solution $y(x)$ of the differential equation

$$x^2 y'(x) + 4y'(x) - x\,y(x) + x = 0$$

for $x \in \mathbb{R}$ with the initial condition $y(0) = 5$ satisfies

☐ $y(\sqrt{5}) = 1$ ☐ $y(\sqrt{5}) = 2$ ☐ $y(\sqrt{5}) = 7$ ☐ $y(\sqrt{5}) = -7$

**Question 4:** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined as

$$f(x,y,z) = -2xy^3z^4 + 2x^2y^2 - 4$$

and consider the point $\boldsymbol{p} = (1,-1,1)$. Since $f(\boldsymbol{p}) = 0$ and $\frac{\partial f}{\partial y}(\boldsymbol{p}) \neq 0$, the equation $f(x,y,z) = 0$ defines on a neighborhood of $(x,z) = (1,1)$ a function $y = g(x,z)$ which satisfies $g(1,1) = -1$ and $f(x, g(x,z), z) = 0$. Moreover

☐ $\frac{\partial g}{\partial z}(1,1) = 1$ ☐ $\frac{\partial g}{\partial z}(1,1) = -\frac{4}{5}$ ☐ $\frac{\partial g}{\partial z}(1,1) = \frac{4}{5}$ ☐ $\frac{\partial g}{\partial z}(1,1) = 4$

**Question 5:** Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25 \text{ and } y \geq 0\}$ and let $f : D \to \mathbb{R}$ be the function defined as

$$f(x,y) = 3y - 4x\,.$$

Then the global maximum value $M = \max_{(x,y)\in D} f(x,y)$ of $f$ on $D$ and the global minimum value $m = \min_{(x,y)\in D} f(x,y)$ of $f$ on $D$ satisfy

- ☐ $M = 25$ and $m = -25$
- ☐ $M = 20$ and $m = -25$
- ☐ $M = 25$ and $m = -20$
- ☐ $M = 20$ and $m = -20$

**Question 6:** Let $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\,,\ y \leq 0\,,\ x \leq 0\}$. Then the integral

$$\iint_D x\,y^3\,dx\,dy$$

is equal to

☐ $\frac{31}{20}$ ☐ $-\frac{21}{8}$ ☐ $\frac{3\pi}{4}$ ☐ $\frac{21}{8}$

**Question 7:** Let $\boldsymbol{h} : \mathbb{R}^2 \to \mathbb{R}^3$ the function be defined as

$$\boldsymbol{h}(u,v) = \left(u^2,\ ve^{-u},\ e^{-2v}\right)^{\mathrm{T}}$$

and let $g : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto g(x,y,z)$, be a function of class $C^1(\mathbb{R}^3)$. Then the partial derivative with respect to $v$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined as

$$f(u,v) = g\big(\boldsymbol{h}(u,v)\big)\,,$$

evaluated at the point $(u,v) = (1,0)$ is equal to

- ☐ $\frac{\partial f}{\partial v}(1,0) = e^{-1}\frac{\partial g}{\partial y}(1,0,1) - 2\frac{\partial g}{\partial z}(1,0,1)$
- ☐ $\frac{\partial f}{\partial v}(1,0) = \frac{\partial g}{\partial y}(1,0,1)$
- ☐ $\frac{\partial f}{\partial v}(1,0) = \frac{\partial g}{\partial x}(1,0,1) - e^{-1}\frac{\partial g}{\partial y}(1,0,1)$
- ☐ $\frac{\partial f}{\partial v}(1,0) = e^{-1}\frac{\partial g}{\partial x}(1,0,1) - e^{-1}\frac{\partial g}{\partial y}(1,0,1)$

**Question 8:** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined as

$$f(x,y,z) = (x^2 + y)z + z^2 - z\,.$$

Then a vector $\boldsymbol{v}$ perpendicular to the level surface of $f$ through the point $(2,0,-1)$ is

☐ $\boldsymbol{v} = \left(1,\ -4,\ 1\right)^{\mathrm{T}}$ ☐ $\boldsymbol{v} = \left(4,\ 1,\ -1\right)^{\mathrm{T}}$

☐ $\boldsymbol{v} = \left(-1,\ -4,\ 4\right)^{\mathrm{T}}$ ☐ $\boldsymbol{v} = \left(-4,\ 1,\ 1\right)^{\mathrm{T}}$

**Question 9:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

$$f(x, y) = 3y - x^2 .$$

The minimum value of $f$ subject to the constraint $g(x, y) = 4x^2 + 9y^2 - 36 = 0$ is

☐ $-6$ ☐ $-9$ ☐ $-10$ ☐ $6$

**Question 10:** Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined as

$$\boldsymbol{f}(x, y) = \left(e^{xy},\ \cos(xy),\ \arctan(x - 2y)\right)^{\mathrm{T}} .$$

Then the Jacobian matrix $J_{\boldsymbol{f}}(x, y)$ of $\boldsymbol{f}$ evaluated at the point $\boldsymbol{p} = (2, 1)$ is

☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} e^2 & 2e^2 \\ -\sin(2) & -2\sin(2) \\ 1 & -2 \end{pmatrix}$ ☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} e^2 & 2e^2 \\ -\sin(2) & -2\sin(2) \\ 1 & 2 \end{pmatrix}$

☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} e^2 & -\sin(2) & 1 \\ 2e^2 & -2\sin(2) & 2 \end{pmatrix}$ ☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} e^2 & -\sin(2) & 1 \\ 2e^2 & -2\sin(2) & -2 \end{pmatrix}$

**Question 11:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let

$$I = \int_{-1}^{1} \left( \int_{0}^{1-|x|} f(x, y)\, dy \right) dx .$$

Then

☐ $I = \int_{0}^{1} \left( \int_{0}^{1-|y|} f(x, y)\, dx \right) dy$

☐ $I = \int_{-1}^{1} \left( \int_{0}^{1-|y|} f(x, y)\, dx \right) dy$

☐ $I = \int_{-1}^{1} \left( \int_{1-y}^{1+y} f(x, y)\, dx \right) dy$

☐ $I = \int_{0}^{1} \left( \int_{y-1}^{1-y} f(x, y)\, dx \right) dy$

**Question 12:** Let $F : (0, \infty) \to \mathbb{R}$ be the function defined as

$$F(t) = \int_{t}^{t^2} \frac{e^{xt}}{x}\, dx .$$

Then we have

☐ $F'(1) = 3e$ ☐ $F'(1) = 1$ ☐ $F'(1) = 0$ ☐ $F'(1) = e$

**Question 13:** The solution $u(t)$ of the differential equation

$$u''(t) - u'(t) - 2u(t) = 4t - 2$$

for $t \in \mathbb{R}$ with the initial conditions $u(0) = 0$ and $u'(0) = 3$ satisfies

☐ $u(1) = e^2 - 3e^{-1}$ ☐ $u(1) = -2e^{-2} - 2e + 2$

☐ $u(1) = e - e^{-2}$ ☐ $u(1) = e^2 - e^{-1}$

**Question 14 :** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = x^3 + y^2 - 2x - y$$

and consider the point $\boldsymbol{p} = (1,-1)$. Then the equation of the tangent plane to the graph of $f$ at $\big(\boldsymbol{p}\,,\, f(\boldsymbol{p})\big)$ is

- ☐ $z + 2x + y + 1 = 0$
- ☐ $z - x + 3y + 3 = 0$
- ☐ $z + 2x + y - 2 = 0$
- ☐ $z - x + 3y - 1 = 0$

**Question 15 :** Let $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^3$ be a curve defined as

$$\boldsymbol{\gamma}(t) = \big(t\cos(t)\,,\, t\sin(t)\,,\, t\big)^{\mathrm{T}}.$$

The length of the arc of $\boldsymbol{\gamma}$ from the point $A = (0,0,0)$ to the point $B = (2\pi, 0, 2\pi)$ is equal to

- ☐ $\displaystyle\int_0^{2\pi} \sqrt{2+t^2}\, dt$
- ☐ $\displaystyle\int_0^{\pi} \sqrt{2+t^2}\, dt$
- ☐ $\displaystyle\int_0^{\pi} (2+t^2)\, dt$
- ☐ $\displaystyle\int_0^{2\pi} 2t^2\, dt$

**Question 16 :** The second order Taylor polynomial of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \sin(xy)$$

at the point $(0,1)$ is

- ☐ $p_2(x,y) = x + y + 2(x-1)y$
- ☐ $p_2(x,y) = x - x^2 - (y-1)^2$
- ☐ $p_2(x,y) = 1 + x + y + 2(x-1)y$
- ☐ $p_2(x,y) = x + x(y-1)$

**Question 17 :** Let $D = \big\{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\big\}$ and $f : D \to \mathbb{R}$ be the function defined as

$$f(x,y) = \ln(x^2 + y)\,.$$

Then the directional derivative of $f$ at the point $(2,1)$ in the direction of the vector $\boldsymbol{e} = \left(\frac{3}{5}\,,\,\frac{4}{5}\right)^{\mathrm{T}}$ is equal to

- ☐ $\dfrac{16}{25}$
- ☐ $-\dfrac{16}{25}$
- ☐ $\dfrac{85}{29}$
- ☐ $-\dfrac{85}{29}$

**Question 18 :** Consider $D = (0,\infty) \times (0,\infty)$ and the function $f : D \to \mathbb{R}$ defined as

$$f(x,y) = \big|\ln(x)\ln(y)\big|\,.$$

Then

- ☐ $f$ is bounded on $D$
- ☐ $f$ is not continuous at $(1,1)$
- ☐ the partial derivatives of $f$ exist at $(1,1)$
- ☐ the partial derivative of $f$ with respect to $y$ does not exist at $(1,1)$

## Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

**Question 19 :** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$. Then, for all $\boldsymbol{p} \in \mathbb{R}^2$ we have

$$\frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{p}) = \frac{\partial^2 f}{\partial y \partial x}(\boldsymbol{p}) .$$

☐ TRUE ☐ FALSE

**Question 20 :** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$ and let $\boldsymbol{p} \in \mathbb{R}^2$. If $\boldsymbol{p}$ is a stationary point of $f$ and if the determinant of the Hessian matrix $\text{Hess}_f(\boldsymbol{p})$ is strictly positive, then $f$ has a local minimum at $\boldsymbol{p}$.

☐ TRUE ☐ FALSE

**Question 21 :** Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Then

$$0 \leq \iint_D \frac{x^2}{x^2 + y^4} \, dx \, dy \leq \pi .$$

☐ TRUE ☐ FALSE

**Question 22 :** The set $D = \{(x, y, z) \in \mathbb{R}^3 : x > 0 \, , \; y > 0 \, , \; z = 0\}$ is open.

☐ TRUE ☐ FALSE

**Question 23 :** Consider a function $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^n$ is a bounded set. If $f$ is continuous on $D$, then $f$ has an global maximum on $D$.

☐ TRUE ☐ FALSE

**Question 24 :** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$ and let $\boldsymbol{p} = (x_0, y_0) \in \mathbb{R}^2$. If $f$ has a local maximum at $\boldsymbol{p}$, then

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x_0 + h, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)}{h} = 0 .$$

☐ TRUE ☐ FALSE

**Question 25 :** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on a closed and bounded set $D \subset \mathbb{R}^2$ and let $\widetilde{D} \subset \mathbb{R}^2$ be a closed and bounded set. If $G : \widetilde{D} \to D$ is a bijective function of class $C^1$ and $J_G(x, y)$ is the Jacobian matrix of $G$, then we have

$$\iint_D f(u, v) \, du \, dv = \iint_{\widetilde{D}} f\big(G(x, y)\big) \Big| \det\big(J_G(x, y)\big) \Big| \, dx \, dy \, ,$$

assuming both integrals exist.

☐ TRUE ☐ FALSE

**Question 26 :** Consider a function $f : \mathbb{R}^3 \to \mathbb{R}$. If $f$ is differentiable at every point of $\mathbb{R}^3$, then $f$ is of class $C^1(\mathbb{R}^3)$.

☐ TRUE ☐ FALSE

**Question 27 :** Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,0) = 1$. If for all fixed $a \in \mathbb{R}$ we have

$$\lim_{x \to 0} f(x, ax) = 1 \,,$$

then $f$ is continuous at $(0,0)$.

☐ TRUE ☐ FALSE

**Question 28 :** Let $p, q : I \to \mathbb{R}$ be two continuous functions defined on the open interval $I \subset \mathbb{R}$ and let $L(u) = u'' + p\,u' + q\,u$. If $u_h$ is a solution of the differential equation $L(u) = 0$ on $I$ and $u_p$ is a solution of the differential equation $L(u) = g$ on $I$, where $g(t) = \cos(t^2)$, then $3u_p + u_h$ is a solution of the differential equation $L(u) = 3g$ on $I$.

☐ TRUE ☐ FALSE