

Duration: 144 minutes

Analysis II

Exam

Common part

Spring 2015

Answers

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: The solution $u(t)$ of the differential equation

$$u''(t) - 4u'(t) + 5u(t) = 8\sin(t)$$

for $t \in \mathbb{R}$ with the initial conditions $u(0) = 2$ and $u'(0) = 5$ is

☐ $u(t) = \sin(t)(4e^{2t} - 1) + \cos(t)(e^{2t} + 1)$

☒ $u(t) = \sin(t)(2e^{2t} + 1) + \cos(t)(e^{2t} + 1)$

☐ $u(t) = \sin(t)(2e^{2t} + 1) - \cos(t)(e^{2t} + 1)$

☐ $u(t) = -\sin(t)(2e^{2t} + 1) + \cos(t)(e^{2t} + 1)$

Question 2: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function defined as

$$f(x, y, z) = 2x^2y^3z^4 + 2x^3y^2 - 3y^2z - 1$$

and consider the point $\mathbf{p} = (1, 1, 1)$. Since $f(\mathbf{p}) = 0$ and $\frac{\partial f}{\partial x}(\mathbf{p}) \neq 0$, the equation $f(x, y, z) = 0$ defines on a neighborhood of $(y, z) = (1, 1)$ a function $x = g(y, z)$ which satisfies $g(1, 1) = 1$ and $f(g(y, z), y, z) = 0$. Moreover

☐ $\frac{\partial g}{\partial z}(1, 1) = \frac{1}{2}$

☐ $\frac{\partial g}{\partial z}(1, 1) = -\frac{4}{5}$

☒ $\frac{\partial g}{\partial z}(1, 1) = -\frac{1}{2}$

☐ $\frac{\partial g}{\partial z}(1, 1) = -2$

Question 3: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y, z) = 2x^2 + 2y^2 - z^2 + 2x + 2y + 1.$$

Then the point $\mathbf{p} = (-\frac{1}{2}, -\frac{1}{2}, 0)$

☐ is a local maximum of f

☐ is a local minimum of f

☒ is a saddle point of f

☐ is not a stationary point of f

Question 4: The second order Taylor polynomial of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as

$$f(x, y) = e^{x^2+y-1}$$

at the point $(1, 0)$ is

☐ $p_2(x, y) = 1 + 2(x - 1) + y + 6(x - 1)^2 + 4(x - 1)y + y^2$

☐ $p_2(x, y) = -1 + 2(x - 1) + y + 3(x - 1)^2 + 2(x - 1)y + \frac{1}{2}y^2$

☒ $p_2(x, y) = 1 + 2(x - 1) + y + 3(x - 1)^2 + 2(x - 1)y + \frac{1}{2}y^2$

☐ $p_2(x, y) = 1 + 2x + y + 3x^2 + 2xy + \frac{1}{2}y^2$

Question 5: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = \begin{cases} x + y + xy \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then

☐ $\frac{\partial f}{\partial x}(0, 0)$ does not exist

☐ f is of class $C^1(\mathbb{R}^2)$

☐ f is not continuous at $(0, 0)$

☒ f is differentiable at $(0, 0)$

Question 6: The solution $y(x)$ of the differential equation

$$(x^2 + 9)y'(x) + xy(x) - x(y(x))^2 = 0$$

for $x \in \mathbb{R}$ with the initial condition $y(0) = \frac{1}{4}$ satisfies

☒ $y(4) = \frac{1}{6}$

☐ $y(4) = 1$

☐ $y(4) = -\frac{1}{4}$

☐ $y(4) = 6$

Question 7: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = x - y - x^2 + y^3$$

and consider the point $\mathbf{p} = (-2, 1)$. Then the equation of the tangent plane to the graph of f at $(\mathbf{p}, f(\mathbf{p}))$ is

☐ $z - 5x - 2y + 6 = 0$

☒ $z - 5x - 2y - 2 = 0$

☐ $z - 2x - 5y + 7 = 0$

☐ $z - 5x - 2y - 8 = 0$

Question 8: Let $D = \{(x, y) \in \mathbb{R}^2 : x > 1 \text{ and } y > -1\}$ and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = \ln(x^2 + y).$$

Then a vector $\mathbf{v}$ in the perpendicular direction to the level curve of f passing through the point $(2, 0)$ at the point $(2, 0)$ is

☐ $\mathbf{v} = (-4, 1)^T$

☒ $\mathbf{v} = (4, 1)^T$

☐ $\mathbf{v} = \left(-\frac{1}{4}, -1\right)^T$

☐ $\mathbf{v} = (1, -4)^T$

Question 9: Let $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the function defined as

$$\mathbf{f}(x, y, z) = \left(\cos(xz), \sin(y - z)\right)^T.$$

Then the Jacobian matrix $J_{\mathbf{f}}(x, y, z)$ of $\mathbf{f}$ evaluated at the point $\mathbf{p} = (1, \frac{\pi}{2}, \frac{\pi}{2})$ is

☒ $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} -\frac{\pi}{2} & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

☐ $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} -\frac{\pi}{2} & 0 \\ 0 & 1 \end{pmatrix}$

☐ $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} -\frac{\pi}{2} \\ 1 \\ 0 \end{pmatrix}$

☐ $J_{\mathbf{f}}(\mathbf{p}) = \begin{pmatrix} -\frac{\pi}{2} & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$

Question 10: Let $D = \{(x, y) \in \mathbb{R}^2 : 4 \leq x^2 + y^2 \leq 16, y \geq 0, x \leq 0\}$. Then the integral

$$\iint_D xy \, dx \, dy$$

is equal to

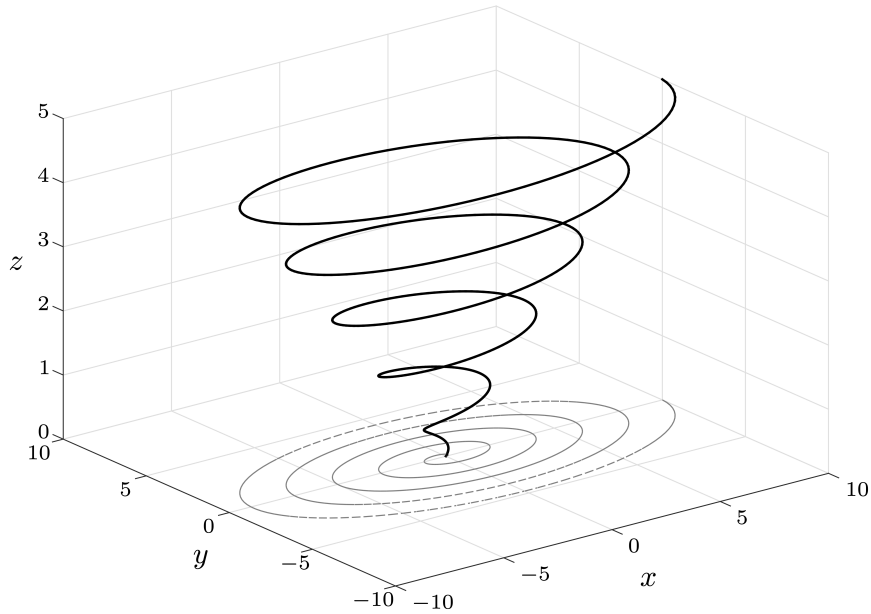
☐ 0

☐ 30

☐ 3π

☒ -30

Question 11: Let a curve with its projection on the x - y plane be illustrated in the following figure:



Which of the following could be a parameterization of the curve?

☐ $(x, y, z) = (t \cos(2\pi t), t \sin(2\pi t), t)$ with $t \in [0, 5]$

☐ $(x, y, z) = (t \cos(2\pi t), 2t \sin(2\pi t), t)$ with $t \in [0, 5]$

☐ $(x, y, z) = (2 \cos(2\pi t), \sin(2\pi t), t)$ with $t \in [0, 5]$

☒ $(x, y, z) = (2t \cos(2\pi t), t \sin(2\pi t), t)$ with $t \in [0, 5]$

Question 12: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function and let

$$I = \int_{-1}^1 \left(\int_{x^2}^1 f(x, y) dy \right) dx .$$

Then

☒ $I = \int_0^1 \left(\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx \right) dy$

☐ $I = \int_0^1 \left(\int_0^{\sqrt{y}} f(x, y) dx \right) dy$

☐ $I = \int_{-1}^1 \left(\int_{-y^2}^{y^2} f(x, y) dx \right) dy$

☐ $I = \int_{-1}^1 \left(\int_0^{\sqrt{y}} f(x, y) dx \right) dy$

Question 13: Let $\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the function defined as

$$\mathbf{h}(u, v) = \left(-u(1 - 2v), u^2(1 - v), uv \right)^T$$

and let $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto g(x, y, z)$, be a function of class $C^1(\mathbb{R}^3)$. Then the partial derivative with respect to v of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined as

$$f(u, v) = g(\mathbf{h}(u, v)) ,$$

evaluated at the point $(u, v) = (1, 0)$ is equal to

☐ $\frac{\partial f}{\partial v}(1, 0) = \frac{\partial g}{\partial x}(-1, 1, 0) - 2\frac{\partial g}{\partial y}(-1, 1, 0) + \frac{\partial g}{\partial z}(-1, 1, 0)$

☐ $\frac{\partial f}{\partial v}(1, 0) = 2\frac{\partial g}{\partial x}(1, 0, 0) - \frac{\partial g}{\partial y}(1, 0, 0) + \frac{\partial g}{\partial z}(1, 0, 0)$

☒ $\frac{\partial f}{\partial v}(1, 0) = 2\frac{\partial g}{\partial x}(-1, 1, 0) - \frac{\partial g}{\partial y}(-1, 1, 0) + \frac{\partial g}{\partial z}(-1, 1, 0)$

☐ $\frac{\partial f}{\partial v}(1, 0) = -\frac{\partial g}{\partial x}(-1, 1, 0) + 2\frac{\partial g}{\partial y}(-1, 1, 0)$

Question 14: Consider $D = \mathbb{R}^2 \setminus \{(0, 0)\}$ and the function $f : D \rightarrow \mathbb{R}$ defined as

$$f(x, y) = \frac{xy^2}{(x^2 + y^4)^{3/2}} .$$

Then

☒ $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist

☐ $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = y$

☐ $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$

☐ $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 1$

Question 15: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = xy.$$

The maximum value of f subject to the constraint $g(x, y) = 2x^2 + y^2 - 4 = 0$ is

☐ $-\sqrt{2}$

☐ 1

☐ 0

☒ $\sqrt{2}$

Question 16: Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } x \geq 0\}$ and let $f : D \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) = y + 2x.$$

Then the global maximum value $M = \max_{(x, y) \in D} f(x, y)$ of f on D and the global minimum value $m = \min_{(x, y) \in D} f(x, y)$ of f on D satisfy

☒ $M = \sqrt{5}$ and $m = -1$

☐ $M = \sqrt{5}$ and $m = -\sqrt{5}$

☐ $M = 1$ and $m = -1$

☐ $M = \sqrt{5}$ and $m = 0$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 17: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class $C^1(\mathbb{R}^2)$. Then the directional derivative of f at $(0, 0)$ in the direction of the vector $\mathbf{v} = (1, 1)^T$ is equal to the limit:

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - f(0,0)}{\sqrt{h^2 + k^2}}.$$

☐ TRUE ☒ FALSE

Question 18: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function. If f is differentiable at every point of $\mathbb{R}^2$, then f is of class $C^1(\mathbb{R}^2)$.

☐ TRUE ☒ FALSE

Question 19: Let $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^m \rightarrow \mathbb{R}^k$ be two functions of class C^1 . Then the function $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is of class C^1 and for all $\mathbf{p} \in \mathbb{R}^n$ the Jacobian matrices of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ satisfy

$$J_{\mathbf{h}}(\mathbf{p}) = J_{\mathbf{g}}(\mathbf{f}(\mathbf{p}))J_{\mathbf{f}}(\mathbf{p}).$$

☒ TRUE ☐ FALSE

Question 20: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function of class $C^2(\mathbb{R}^3)$ and let $\mathbf{p} \in \mathbb{R}^3$. If $\mathbf{p}$ is a stationary point of f and if the determinant of the Hessian matrix $\text{Hess}_f(\mathbf{p})$ is strictly negative, then $\mathbf{p}$ is a local maximum of f .

☐ TRUE ☒ FALSE

Question 21: Let A and B be two subsets of $\mathbb{R}^n$ and let $\mathbf{f} : A \rightarrow B$ be a bijective function such that $\mathbf{f}$ and $\mathbf{f}$ are of class C^1 . Then for all $\mathbf{p} \in A$ we have $\det(J_{\mathbf{f}}(\mathbf{p})) \neq 0$.

☒ TRUE ☐ FALSE

Question 22: Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function of class $C^1(\mathbb{R}^n)$ and let $\mathbf{p} \in \mathbb{R}^n$. If $\mathbf{p}$ is a local extremum of f , then $\mathbf{p}$ is a stationary point of f .

☒ TRUE ☐ FALSE

Question 23: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function on a closed and bounded set $D \subset \mathbb{R}^2$ and let $\tilde{D} \subset \mathbb{R}^2$ be a closed and bounded set. If $G : \tilde{D} \rightarrow D$ is a bijective function of class C^1 and $J_G(u, v)$ is the Jacobian matrix of G , then we have

$$\iint_D f(x, y) \, dx \, dy = \iint_{\tilde{D}} f(G(u, v)) \left| \det(J_G(u, v)) \right| \, du \, dv,$$

assuming both integrals exist.

☒ TRUE ☐ FALSE

Question 24: Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0\}$. Then

$$\iint_D \frac{\tan(y)}{x^2 + y^2 + 1} dx dy \geq 1.$$

☐ TRUE ☒ FALSE

Question 25: Let $\mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field of class C^2 . Then $\nabla(\operatorname{div} \mathbf{v}) = 0$.

☐ TRUE ☒ FALSE

Question 26: Consider a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f(0, 0) = 1$. If for all fixed $\varphi \in [0, 2\pi[$ we have

$$\lim_{r \rightarrow 0} f(r \cos(\varphi), r \sin(\varphi)) = 1,$$

then f is continuous at $(0, 0)$.

☐ TRUE ☒ FALSE

Question 27: Let $p, q, g : I \rightarrow \mathbb{R}$ be three continuous functions defined on the open interval $I \subset \mathbb{R}$ and let $L(u) = u'' + pu' + qu$. If u_h is a solution of the differential equation $L(u) = 0$ on I and u_p is a solution of the differential equation $L(u) = g$ on I , then $u_p + \frac{1}{2}u_h$ is a solution of the differential equation $L(u) = g$ on I .

☒ TRUE ☐ FALSE

Question 28: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto f(x, y, z)$, be a function that is differentiable at a point $\mathbf{p} \in \mathbb{R}^3$. Then the vector $\left(-\frac{\partial f}{\partial x}(\mathbf{p}), -\frac{\partial f}{\partial y}(\mathbf{p}), -\frac{\partial f}{\partial z}(\mathbf{p}), 1\right)^T$ is perpendicular to the hyperplane tangent to the graph of f at the point $(\mathbf{p}, f(\mathbf{p}))$.

☒ TRUE ☐ FALSE

Question 29: The set $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } x \neq 0\}$ is closed.

☐ TRUE ☒ FALSE

Question 30: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$. Then, for all $\mathbf{p} \in \mathbb{R}^2$ we have

$$\frac{\partial^2 f}{\partial x^2}(\mathbf{p}) = \frac{\partial^2 f}{\partial y^2}(\mathbf{p}).$$

☐ TRUE ☒ FALSE

Question 31: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$, be a function of class $C^2(\mathbb{R}^2)$. Then for all points $(x, y) \in \mathbb{R}^2$ we have

$$\lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(x, y+h) - \frac{\partial f}{\partial x}(x, y)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial y}(x+h, y) - \frac{\partial f}{\partial y}(x, y)}{h}.$$

☒ TRUE ☐ FALSE

Question 32: Let $f : D \rightarrow \mathbb{R}$ be a function, with $D \subset \mathbb{R}^n$ a closed and bounded set. If f does not have an global maximum on D , then f is not continuous on D .

☒ TRUE ☐ FALSE