Duration : 144 minutes

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Analysis II
Exam

Common part

Spring 2015

Questions

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The solution $u(t)$ of the differential equation

$$u''(t) - 4u'(t) + 5u(t) = 8\sin(t)$$

for $t \in \mathbb{R}$ with the initial conditions $u(0) = 2$ and $u'(0) = 5$ is

- ☐ $u(t) = \sin(t)\big(4e^{2t} - 1\big) + \cos(t)\big(e^{2t} + 1\big)$
- ☐ $u(t) = \sin(t)\big(2e^{2t} + 1\big) + \cos(t)\big(e^{2t} + 1\big)$
- ☐ $u(t) = \sin(t)\big(2e^{2t} + 1\big) - \cos(t)\big(e^{2t} + 1\big)$
- ☐ $u(t) = -\sin(t)\big(2e^{2t} + 1\big) + \cos(t)\big(e^{2t} + 1\big)$

Question 2 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined as

$$f(x, y, z) = 2x^2y^3z^4 + 2x^3y^2 - 3y^2z - 1$$

and consider the point $\boldsymbol{p} = (1, 1, 1)$. Since $f(\boldsymbol{p}) = 0$ and $\frac{\partial f}{\partial x}(\boldsymbol{p}) \neq 0$, the equation $f\big(x, y, z\big) = 0$ defines on a neighborhood of $(y, z) = (1, 1)$ a function $x = g(y, z)$ which satisfies $g(1, 1) = 1$ and $f\big(g(y, z), y, z\big) = 0$. Moreover

☐ $\frac{\partial g}{\partial z}(1,1) = \frac{1}{2}$ ☐ $\frac{\partial g}{\partial z}(1,1) = -\frac{4}{5}$ ☐ $\frac{\partial g}{\partial z}(1,1) = -\frac{1}{2}$ ☐ $\frac{\partial g}{\partial z}(1,1) = -2$

Question 3 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x, y, z) = 2x^2 + 2y^2 - z^2 + 2x + 2y + 1\,.$$

Then the point $\boldsymbol{p} = \big(-\frac{1}{2}, -\frac{1}{2}, 0\big)$

- ☐ is a local maximum of f
- ☐ is a local minimum of f
- ☐ is a saddle point of f
- ☐ is not a stationary point of f

Question 4 : The second order Taylor polynomial of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = e^{x^2+y-1}$$

at the point $(1, 0)$ is

- ☐ $p_2(x, y) = 1 + 2(x - 1) + y + 6(x - 1)^2 + 4(x - 1)y + y^2$
- ☐ $p_2(x, y) = -1 + 2(x - 1) + y + 3(x - 1)^2 + 2(x - 1)y + \frac{1}{2}\,y^2$
- ☐ $p_2(x, y) = 1 + 2(x - 1) + y + 3(x - 1)^2 + 2(x - 1)y + \frac{1}{2}\,y^2$
- ☐ $p_2(x, y) = 1 + 2x + y + 3x^2 + 2xy + \frac{1}{2}\,y^2$

Question 5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = \begin{cases} x + y + xy\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- ☐ $\dfrac{\partial f}{\partial x}(0,0)$ does not exist
- ☐ f is of class $C^1(\mathbb{R}^2)$
- ☐ f is not continuous at $(0,0)$
- ☐ f is differentiable at $(0,0)$

Question 6: The solution $y(x)$ of the differential equation

$$(x^2+9)\,y'(x) + x\,y(x) - x\big(y(x)\big)^2 = 0$$

for $x \in \mathbb{R}$ with the initial condition $y(0) = \frac{1}{4}$ satisfies

☐ $y(4) = \frac{1}{6}$ ☐ $y(4) = 1$ ☐ $y(4) = -\frac{1}{4}$ ☐ $y(4) = 6$

Question 7: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = x - y - x^2 + y^3$$

and consider the point $\boldsymbol{p} = (-2, 1)$. Then the equation of the tangent plane to the graph of f at $\big(\boldsymbol{p}\,,\, f(\boldsymbol{p})\big)$ is

- ☐ $z - 5x - 2y + 6 = 0$
- ☐ $z - 5x - 2y - 2 = 0$
- ☐ $z - 2x - 5y + 7 = 0$
- ☐ $z - 5x - 2y - 8 = 0$

Question 8: Let $D = \big\{(x,y) \in \mathbb{R}^2 : x > 1 \text{ and } y > -1\big\}$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = \ln\big(x^2 + y\big)\,.$$

Then a vector $\boldsymbol{v}$ in the perpendicular direction to the level curve of f passing through the point $(2,0)$ at the point $(2,0)$ is

- ☐ $\boldsymbol{v} = \big(-4,\, 1\big)^{\mathrm{T}}$
- ☐ $\boldsymbol{v} = \big(4,\, 1\big)^{\mathrm{T}}$
- ☐ $\boldsymbol{v} = \left(-\frac{1}{4}\,,\, -1\right)^{\mathrm{T}}$
- ☐ $\boldsymbol{v} = \big(1,\, -4\big)^{\mathrm{T}}$

Question 9: Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined as

$$\boldsymbol{f}(x,y,z) = \Big(\cos(xz),\, \sin(y-z)\Big)^{\mathrm{T}}.$$

Then the Jacobian matrix $J_{\boldsymbol{f}}(x,y,z)$ of $\boldsymbol{f}$ evaluated at the point $\boldsymbol{p} = \left(1, \frac{\pi}{2}, \frac{\pi}{2}\right)$ is

- ☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} -\frac{\pi}{2} & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$
- ☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} -\frac{\pi}{2} & 0 \\ 0 & 1 \end{pmatrix}$
- ☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} -\frac{\pi}{2} \\ 1 \\ 0 \end{pmatrix}$
- ☐ $J_{\boldsymbol{f}}(\boldsymbol{p}) = \begin{pmatrix} -\frac{\pi}{2} & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$

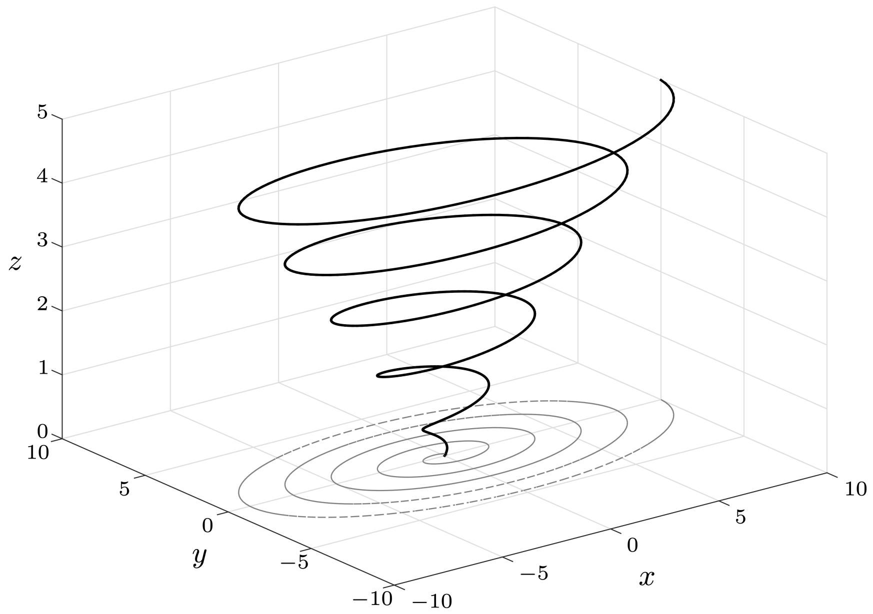
Question 10 : Let $D = \{(x,y) \in \mathbb{R}^2 : 4 \leq x^2 + y^2 \leq 16\,,\ y \geq 0\,,\ x \leq 0\}$. Then the integral

$$\iint_D x\,y\,dx\,dy$$

is equal to

☐ 0 ☐ 30 ☐ 3π ☐ -30

Question 11 : Let a curve with its projection on the x-y plane be illustrated in the following figure:



Which of the following could be a parameterization of the curve?

☐ $(x,y,z) = \big(t\cos(2\pi t),\ t\sin(2\pi t),\ t\big)$ with $t \in [0,5]$

☐ $(x,y,z) = \big(t\cos(2\pi t),\ 2t\sin(2\pi t),\ t\big)$ with $t \in [0,5]$

☐ $(x,y,z) = \big(2\cos(2\pi t),\ \sin(2\pi t),\ t\big)$ with $t \in [0,5]$

☐ $(x,y,z) = \big(2t\cos(2\pi t),\ t\sin(2\pi t),\ t\big)$ with $t \in [0,5]$

Question 12: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let

$$I = \int_{-1}^{1} \left(\int_{x^2}^{1} f(x,y)\, dy \right) dx\,.$$

Then

- ☐ $I = \displaystyle\int_{0}^{1} \left(\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx \right) dy$
- ☐ $I = \displaystyle\int_{0}^{1} \left(\int_{0}^{\sqrt{y}} f(x,y)\, dx \right) dy$
- ☐ $I = \displaystyle\int_{-1}^{1} \left(\int_{-y^2}^{y^2} f(x,y)\, dx \right) dy$
- ☐ $I = \displaystyle\int_{-1}^{1} \left(\int_{0}^{\sqrt{y}} f(x,y)\, dx \right) dy$

Question 13: Let $\boldsymbol{h} : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined as

$$\boldsymbol{h}(u,v) = \Big(-u(1-2v),\, u^2(1-v),\, uv \Big)^{\mathrm{T}}$$

and let $g : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto g(x,y,z)$, be a function of class $C^1(\mathbb{R}^3)$. Then the partial derivative with respect to v of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined as

$$f(u,v) = g\big(\boldsymbol{h}(u,v)\big)\,,$$

evaluated at the point $(u,v) = (1,0)$ is equal to

- ☐ $\dfrac{\partial f}{\partial v}(1,0) = \dfrac{\partial g}{\partial x}(-1,1,0) - 2\dfrac{\partial g}{\partial y}(-1,1,0) + \dfrac{\partial g}{\partial z}(-1,1,0)$
- ☐ $\dfrac{\partial f}{\partial v}(1,0) = 2\dfrac{\partial g}{\partial x}(1,0,0) - \dfrac{\partial g}{\partial y}(1,0,0) + \dfrac{\partial g}{\partial z}(1,0,0)$
- ☐ $\dfrac{\partial f}{\partial v}(1,0) = 2\dfrac{\partial g}{\partial x}(-1,1,0) - \dfrac{\partial g}{\partial y}(-1,1,0) + \dfrac{\partial g}{\partial z}(-1,1,0)$
- ☐ $\dfrac{\partial f}{\partial v}(1,0) = -\dfrac{\partial g}{\partial x}(-1,1,0) + 2\dfrac{\partial g}{\partial y}(-1,1,0)$

Question 14: Consider $D = \mathbb{R}^2 \backslash \big\{(0,0)\big\}$ and the function $f : D \to \mathbb{R}$ defined as

$$f(x,y) = \frac{xy^2}{\big(x^2+y^4\big)^{3/2}}\,.$$

Then

- ☐ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist
- ☐ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = y$
- ☐ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$
- ☐ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 1$

Question 15 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x, y) = xy\,.$$

The maximum value of f subject to the constraint $g(x, y) = 2x^2 + y^2 - 4 = 0$ is

- ☐ $-\sqrt{2}$
- ☐ 1
- ☐ 0
- ☐ $\sqrt{2}$

Question 16 : Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } x \geq 0\}$ and let $f : D \to \mathbb{R}$ be the function defined as

$$f(x, y) = y + 2x\,.$$

Then the global maximum value $M = \max_{(x,y)\in D} f(x, y)$ of f on D and the global minimum value $m = \min_{(x,y)\in D} f(x, y)$ of f on D satisfy

- ☐ $M = \sqrt{5}$ and $m = -1$
- ☐ $M = \sqrt{5}$ and $m = -\sqrt{5}$
- ☐ $M = 1$ and $m = -1$
- ☐ $M = \sqrt{5}$ and $m = 0$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 17 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^1(\mathbb{R}^2)$. Then the directional derivative of f at $(0,0)$ in the direction of the vector $\boldsymbol{v} = \big(1,\, 1\big)^{\mathrm{T}}$ is equal to the limit:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0)}{\sqrt{h^2+k^2}}\,.$$

☐ TRUE ☐ FALSE

Question 18 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If f is differentiable at every point of $\mathbb{R}^2$, then f is of class $C^1(\mathbb{R}^2)$.

☐ TRUE ☐ FALSE

Question 19 : Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^k$ be two functions of class C^1. Then the function $\boldsymbol{h} = \boldsymbol{g} \circ \boldsymbol{f}$ is of class C^1 and for all $\boldsymbol{p} \in \mathbb{R}^n$ the Jacobian matrices of $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$ satisfy

$$J_{\boldsymbol{h}}(\boldsymbol{p}) = J_{\boldsymbol{g}}\big(\boldsymbol{f}(\boldsymbol{p})\big)\, J_{\boldsymbol{f}}(\boldsymbol{p})\,.$$

☐ TRUE ☐ FALSE

Question 20 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class $C^2(\mathbb{R}^3)$ and let $\boldsymbol{p} \in \mathbb{R}^3$. If $\boldsymbol{p}$ is a stationary point of f and if the determinant of the Hessian matrix $\mathrm{Hess}_f(\boldsymbol{p})$ is strictly negative, then $\boldsymbol{p}$ is a local maximum of f.

☐ TRUE ☐ FALSE

Question 21 : Let A and B be two subsets of $\mathbb{R}^n$ and let $\boldsymbol{f} : A \to B$ be a bijective function such that $\boldsymbol{f}$ and $\boldsymbol{f}$ are of class C^1. Then for all $\boldsymbol{p} \in A$ we have $\det\big(J_{\boldsymbol{f}}(\boldsymbol{p})\big) \neq 0$.

☐ TRUE ☐ FALSE

Question 22 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^1(\mathbb{R}^n)$ and let $\boldsymbol{p} \in \mathbb{R}^n$. If $\boldsymbol{p}$ is a local extremum of f, then $\boldsymbol{p}$ is a stationary point of f.

☐ TRUE ☐ FALSE

Question 23 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on a closed and bounded set $D \subset \mathbb{R}^2$ and let $\widetilde{D} \subset \mathbb{R}^2$ be a closed and bounded set. If $G : \widetilde{D} \to D$ is a bijective function of class C^1 and $J_G(u,v)$ is the Jacobian matrix of G, then we have

$$\iint_D f(x,y)\,dx\,dy = \iint_{\widetilde{D}} f\big(G(u,v)\big)\Big|\det\big(J_G(u,v)\big)\Big|\,du\,dv\,,$$

assuming both integrals exist.

☐ TRUE ☐ FALSE

Question 24 : Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ \ x \geq 0\}$. Then

$$\iint_D \frac{\tan(y)}{x^2 + y^2 + 1}\, dx\, dy \ \geq \ 1\,.$$

☐ TRUE ☐ FALSE

Question 25 : Let $\boldsymbol{v} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^2. Then $\nabla(\operatorname{div} \boldsymbol{v}) = 0$.

☐ TRUE ☐ FALSE

Question 26 : Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,0) = 1$. If for all fixed $\varphi \in [0, 2\pi[$ we have

$$\lim_{r \to 0} f\big(r\cos(\varphi), r\sin(\varphi)\big) = 1\,,$$

then f is continuous at $(0,0)$.

☐ TRUE ☐ FALSE

Question 27 : Let $p, q, g : I \to \mathbb{R}$ be three continuous functions defined on the open interval $I \subset \mathbb{R}$ and let $L(u) = u'' + p\,u' + q\,u$. If u_h is a solution of the differential equation $L(u) = 0$ on I and u_p is a solution of the differential equation $L(u) = g$ on I, then $u_p + \frac{1}{2}u_h$ is a solution of the differential equation $L(u) = g$ on I.

☐ TRUE ☐ FALSE

Question 28 : Let $f : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto f(x,y,z)$, be a function that is differentiable at a point $\boldsymbol{p} \in \mathbb{R}^3$. Then the vector $\Big(-\frac{\partial f}{\partial x}(\boldsymbol{p}),\ -\frac{\partial f}{\partial y}(\boldsymbol{p}),\ -\frac{\partial f}{\partial z}(\boldsymbol{p}),\ 1\Big)^{\mathrm{T}}$ is perpendicular to the hyperplane tangent to the graph of f at the point $\big(\boldsymbol{p},\ f(\boldsymbol{p})\big)$.

☐ TRUE ☐ FALSE

Question 29 : The set $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } x \neq 0\}$ is closed.

☐ TRUE ☐ FALSE

Question 30 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^2(\mathbb{R}^2)$. Then, for all $\boldsymbol{p} \in \mathbb{R}^2$ we have

$$\frac{\partial^2 f}{\partial x^2}(\boldsymbol{p}) = \frac{\partial^2 f}{\partial y^2}(\boldsymbol{p})\,.$$

☐ TRUE ☐ FALSE

Question 31 : Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$, be a function of class $C^2(\mathbb{R}^2)$. Then for all points $(x,y) \in \mathbb{R}^2$ we have

$$\lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(x,y+h) - \frac{\partial f}{\partial x}(x,y)}{h} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(x+h,y) - \frac{\partial f}{\partial y}(x,y)}{h}.$$

☐ TRUE ☐ FALSE

Question 32 : Let $f : D \to \mathbb{R}$ be a function, with $D \subset \mathbb{R}^n$ a closed and bounded set. If f does not have an global maximum on D, then f is not continuous on D.

☐ TRUE ☐ FALSE