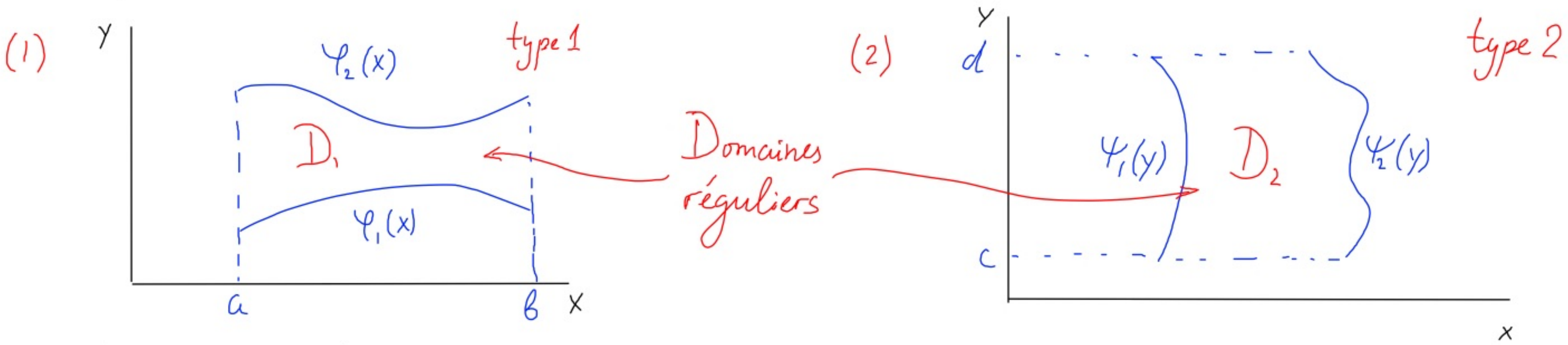
Rappel: Thm de Fubini pour les domaines réguliers dans $\mathbb{R}^2$



$\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ continues, $\varphi_1(x) < \varphi_2(x)$ $\forall x \in]a,b[$

$f : \overline{D}_1 \to \mathbb{R}$ continue. Alors

$$\iint_{D_1} f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx$$

$\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ continues, $\psi_1(y) < \psi_2(y)$ $\forall y \in]c,d[$

$f : \overline{D}_2 \to \mathbb{R}$ continue. Alors

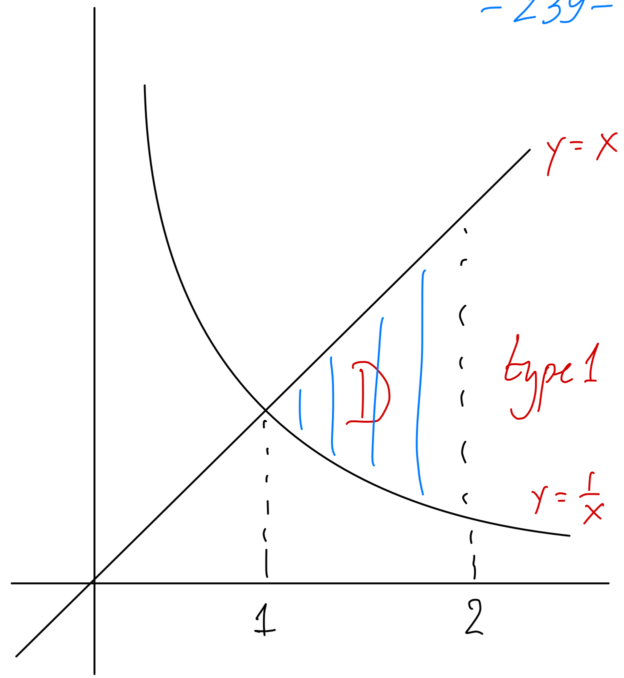
$$\iint_{D_2} f(x,y)\,dx\,dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy .$$

Si le domaine D est plus compliqué, il faut le diviser en réunion des domaines de type 1, 2, et utiliser l'additivité de l'intégrale.

Ex2. $f(x,y) = \dfrac{x^2}{y^2}$ sur $D = \{(x,y) \in \mathbb{R}^2 : 1 < x < 2,\ \frac{1}{x} < y < x\}$

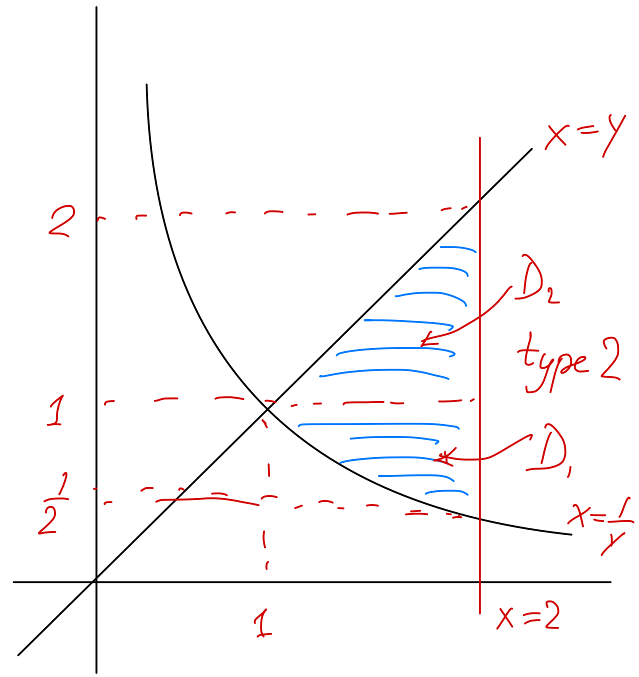
$$\iint_D \frac{x^2}{y^2}\,dx\,dy = \int_1^2 \left(\int_{\frac{1}{x}}^{x} \frac{x^2}{y^2}\,dy \right) dx = \int_1^2 \left(-\frac{x^2}{y} \Big|_{y=\frac{1}{x}}^{x} \right) dx =$$

$$= \int_1^2 (-x + x^3)\,dx = -\frac{1}{2}x^2 + \frac{1}{4}x^4 \Big|_1^2 = -2 + \frac{1}{2} + 4 - \frac{1}{4} = \frac{9}{4}$$



$$\iint_D \frac{x^2}{y^2}\,dx\,dy = \underbrace{\int_{\frac{1}{2}}^{1} \left(\int_{\frac{1}{y}}^{2} \frac{x^2}{y^2}\,dx \right) dy}_{D_1} + \underbrace{\int_1^2 \left(\int_y^2 \frac{x^2}{y^2}\,dx \right) dy}_{D_2} =$$

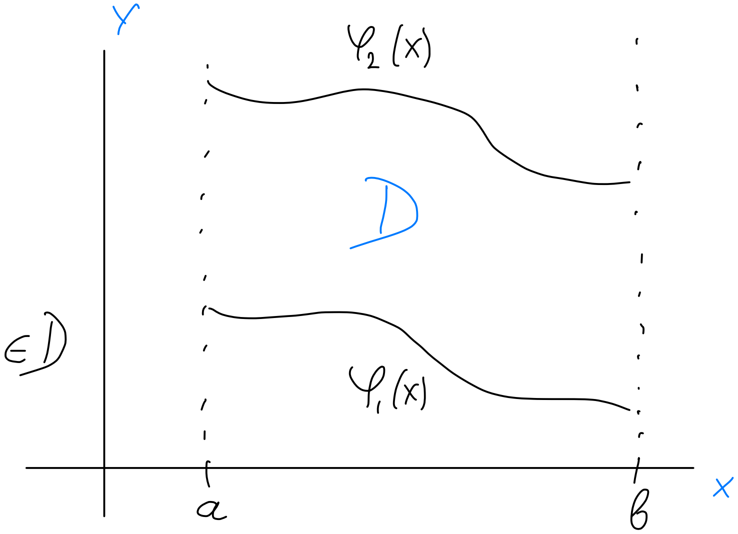
$$= \ldots \quad [\text{Exercice}] = \frac{9}{4}.$$



Thm de Fubini pour les intégrales triples.

Soit $[a,b]$ intervalle, $a<b$; $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ continues

$$\varphi_1(x) < \varphi_2(x) \quad \forall x \in]a,b[.$$



$$D = \{(x,y) \in \mathbb{R}^2 : a<x<b,\ \varphi_1(x)<y<\varphi_2(x)\}$$

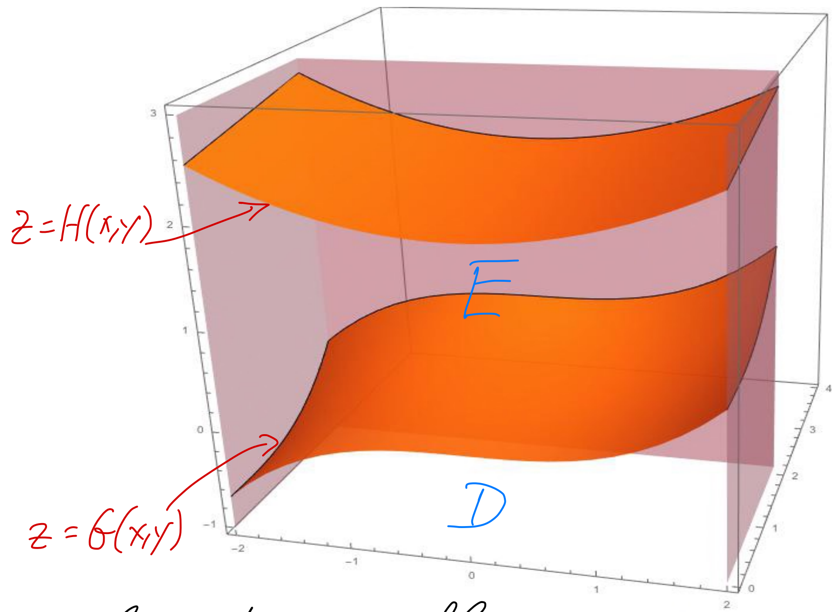
Soient $G, H : \overline{D} \to \mathbb{R}$ continues, $G(x,y) < H(x,y)$ $\forall (x,y) \in D$

$$E = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ G(x,y) < z < H(x,y)\}.$$

$f : \overline{E} \to \mathbb{R}$ continue

Alors f est intégrable sur E et on a :

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} \left(\int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz \right) dy \right) dx$$

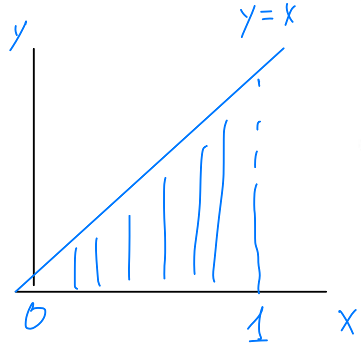


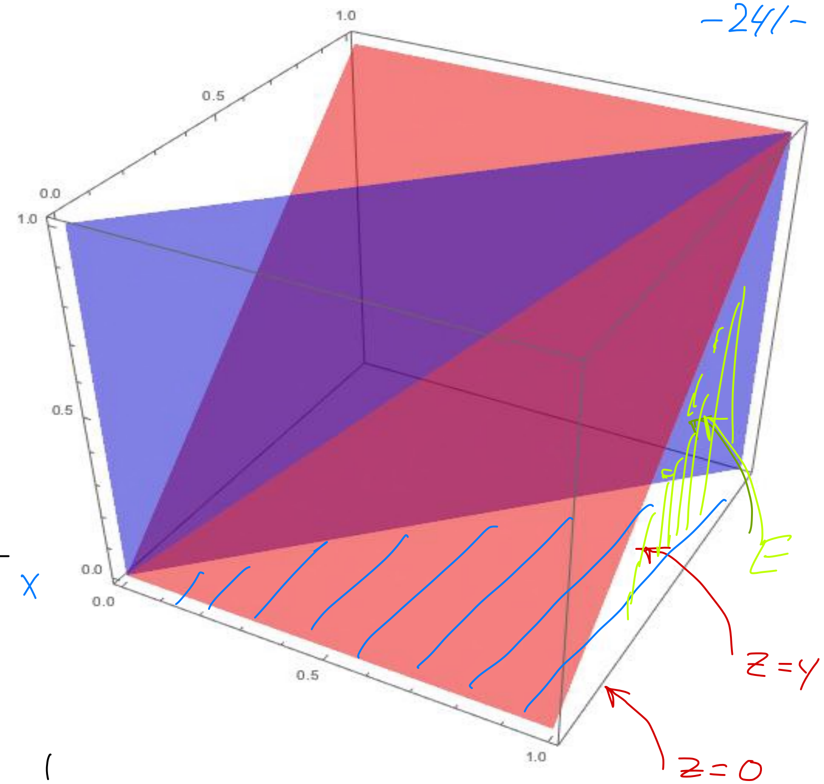
Remarque. Selon la géometrie du domaine, les rôles des variables x, y et z peuvent être échangés.

Ex 3. $E = \{(x,y,z)\in \mathbb{R}^3 : 0<z<y<x<1\}$

$f(x,y,z) = e^{x^3}$

$E = \{(x,y,z)\in \mathbb{R}^3 : 0<x<1;\ 0<y<x;\ 0<z<y\}$





$$\iiint_E e^{x^3}\,dx\,dy\,dz = \int_0^1\left(\int_0^x\left(\int_0^y e^{x^3}dz\right)dy\right)dx =$$

notation

$$= \int_0^1 dx \int_0^x dy \int_0^y e^{x^3} dz = \int_0^1 dx \int_0^x dy\, \left(z e^{x^3}\right)\Big|_0^y =$$

$$= \int_0^1 dx \int_0^x dy\, \left(y e^{x^3}\right) = \int_0^1 dx \cdot \frac{1}{2} y^2 e^{x^3}\Big|_0^x = \int_0^1 dx \left(\frac{1}{2}x^2 e^{x^3}\right) = \frac{1}{6}\int_0^1 e^{x^3} d(x^3) =$$

$$= \frac{1}{6} e^{x^3}\Big|_0^1 = \frac{1}{6}(e-1).$$

Changement d'ordre d'intégration :

$E = \{(x,y,z)\in \mathbb{R}^3 : 0<x<1,\ 0<z<x,\ z<y<x\}$

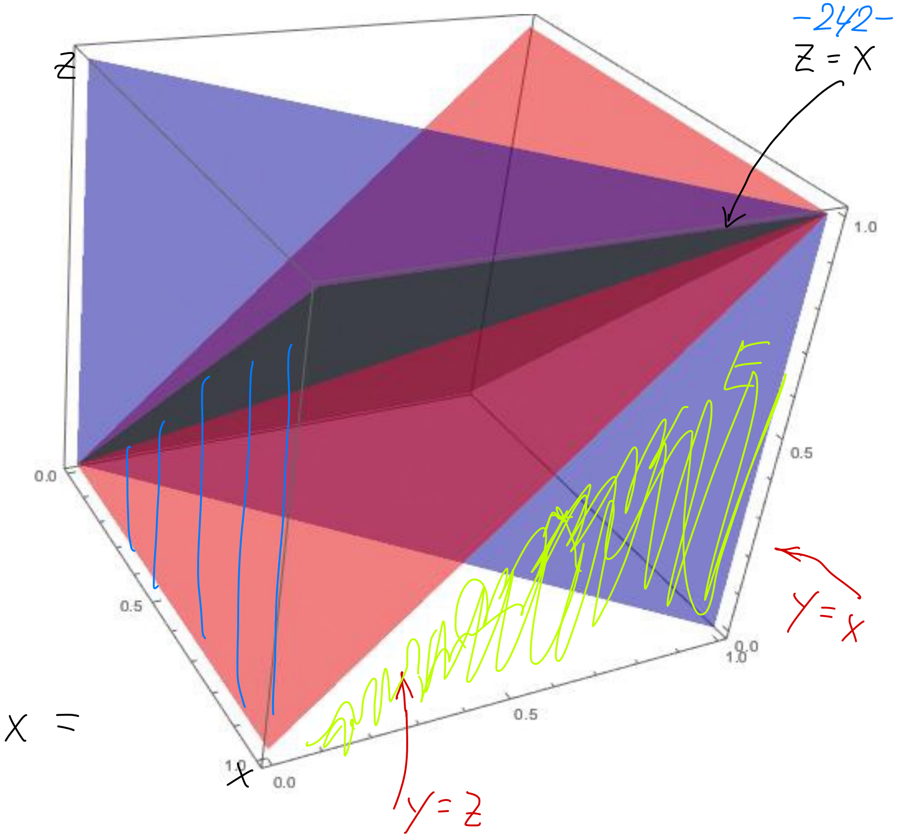
$$E=\{(x,y,z)\in\mathbb{R}^3:\ 0<x<1,\ 0<z<x,\ z<y<x\}$$

$$\iiint_E e^{x^3}\,dx\,dy\,dz=\int_0^1 dx\int_0^x dz\int_z^x e^{x^3}dy=$$

$$=\int_0^1 dx\int_0^x dz\ y\,e^{x^3}\Big|_{y=z}^{x}=\int_0^1 dx\int_0^x dz\,(x-z)e^{x^3}=$$

$$=\int_0^1 dx\,\left(xz-\frac{1}{2}z^2\right)e^{x^3}\Big|_{z=0}^{x}=\int_0^1 dx\,\left(x^2-\frac{1}{2}x^2\right)e^{x^3}=\int_0^1 \frac{1}{2}x^2e^{x^3}dx=$$

$$=\frac{1}{6}\int_0^1 e^{x^3}d(x^3)=\frac{1}{6}e^{x^3}\Big|_0^1=\frac{1}{6}(e-1).$$



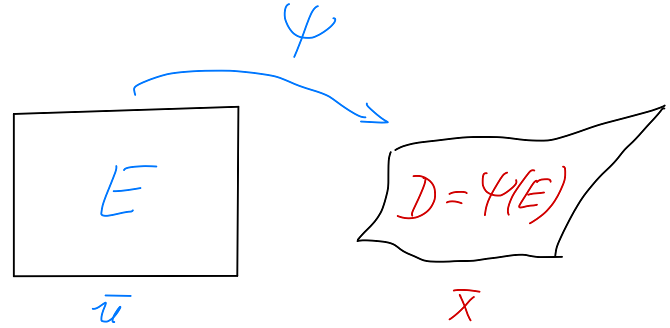
Changement de variables dans l'intégrale multiple

Thm. Soit $E \subset \mathbb{R}^n$ un sous-ensemble ($\bar{E}$ compact) ; $\psi : E \to \mathbb{R}^n$ telle que $\psi \in C^1(E)$ et $\psi : E \to \psi(E)$ est bijective ($J_\psi(\bar{u})$ est inversible $\forall \bar{u} \in E$).

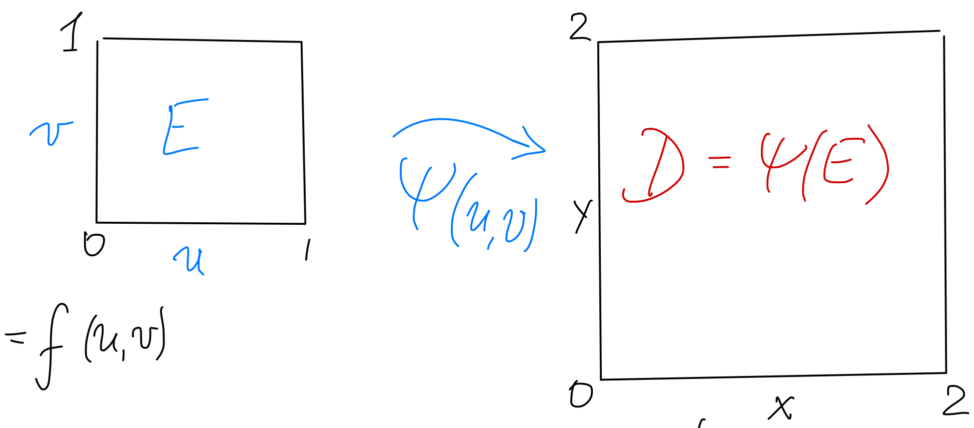
Soit $f : \bar{D} = \overline{\psi(E)} \to \mathbb{R}$ fonction continue.

Alors

$$\boxed{\int_D f(\bar{x})\,d\bar{x} = \int_E f(\psi(\bar{u}))\,|\det J_\psi(\bar{u})|\,d\bar{u}}$$



Ex 0.



$$\psi(u,v) = \begin{pmatrix} 2u \\ 2v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$J_\psi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$|\det J_\psi| = 4 \quad \forall (u,v) \in E$$

$$f(x,y) = 1 = f(u,v)$$

$$\iint_D f(x,y)\,dx\,dy = \iint_E 1 \cdot |\det J_\psi|\,du\,dv = \int_0^1 du \int_0^1 dv \cdot 4 = 4 \cdot 1 \cdot 1 = 4.$$

$$\parallel$$

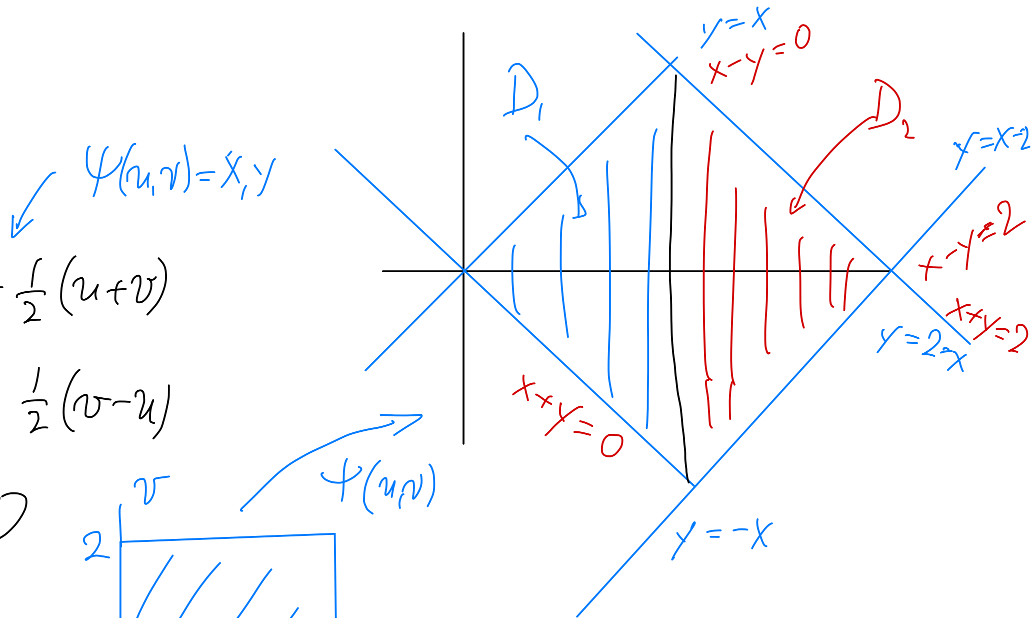
$$\int_0^2 dx \int_0^2 1\,dy = 1 \cdot 2 \cdot 2 = 4$$

Ex 1

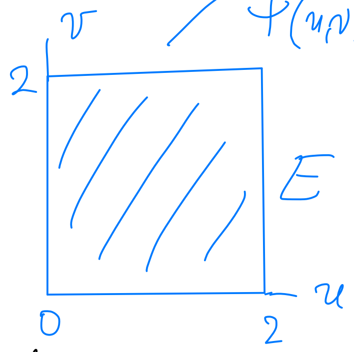
$f(x,y)=x^2$, $D=\{(x,y)\in\mathbb{R}^2 : 0<x<1, -x<y<x\} \cup \{(x,y)\in\mathbb{R}^2 : 1<x<2, x-2<y<2-x\}$.

D_1 $\qquad$ D_2

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 dx \int_{-x}^{x} x^2\,dy + \int_1^2 dx \int_{x-2}^{2-x} x^2\,dy .$$



Plus facile : $\begin{cases} u=x-y \\ v=x+y \end{cases} \Rightarrow \begin{matrix} 0<u<2 \\ 0<v<2 \end{matrix} \Rightarrow \begin{cases} x=\frac{1}{2}(u+v) \\ y=\frac{1}{2}(v-u) \end{cases}$



$$J_{\psi(u,v)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow \det J_{\psi(u,v)} = \frac{1}{4}+\frac{1}{4}=\frac{1}{2}\neq 0$$

sur E

$\Rightarrow \psi : E \to D$ bijective ; $f(x,y)=x^2=\frac{1}{4}(u+v)^2$.

Alors on a : $\displaystyle\iint_D x^2\,dx\,dy = \frac{1}{4}\iint_E (u+v)^2 \underbrace{|\det J_{\psi(u,v)}|}_{\frac{1}{2}}\,du\,dv = \int_0^2 du \int_0^2 dv\, \frac{1}{8}(u+v)^2 =$

$$= \frac{1}{24}\int_0^2 du\, (u+v)^3 \Big|_{\underset{v}{0}}^{2} = \frac{1}{24}\int_0^2 \left((u+2)^3-u^3\right)du = \frac{1}{24}\cdot\frac{1}{4}\left((u+2)^4-u^4\right)\Big|_0^2 = \frac{1}{24}\cdot\frac{1}{4}\left(4^4-2^4-2^4+0\right) =$$

$$= \frac{1}{6}\cdot\frac{1}{\cancel{16}}\cdot\left(\cancel{16}\left(2^4-2\right)\right) = \frac{1}{6}\cdot 14 = \frac{7}{3}.$$

Intégration directe : (exercice)

$$\iint_D x^2\,dx\,dy = \int_0^1 dx \int_{-x}^{x} x^2\,dy + \int_1^2 dx \int_{x-2}^{2-x} x^2\,dy = \int_0^1 dx\,(yx^2)\Big|_{y=-x}^{x} + \int_1^2 dx\,(yx^2)\Big|_{y=x-2}^{2-x} =$$

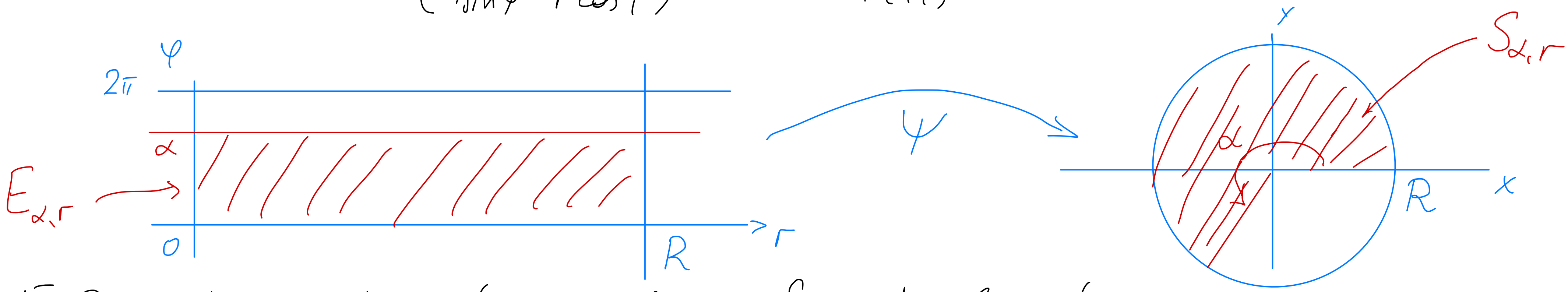
$$= \int_0^1 2x^3\,dx + \int_1^2 (4-2x)x^2\,dx = \frac{1}{2}x^4\Big|_0^1 + \frac{4}{3}x^3\Big|_1^2 - \frac{1}{2}x^4\Big|_1^2 = \frac{1}{2} + \frac{32}{3} - \frac{4}{3} - 8 + \frac{1}{2} = -7 + 10 + \frac{2}{3} - \frac{4}{3} =$$

$$= 3 - \frac{2}{3} = \frac{7}{3}.$$

Application : changement de variables polaires.

$$\psi(r,\varphi) = \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \psi : \mathbb{R}_+ \times [0,2\pi[\xrightarrow[\text{bijective}]{} \mathbb{R}^2 \setminus \{0\}.$$

$$J_\psi(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \Rightarrow \det J_{\psi(r,\varphi)} = r(\cos^2\varphi + \sin^2\varphi) = r.$$



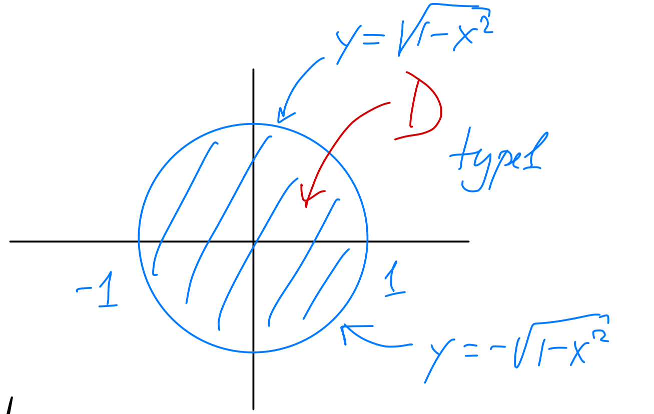
Ex 2. L'aire du secteur circulaire $S_{\alpha,r}$ d'angle α et rayon R

$$A_{\alpha,R} = \iint_{S_{\alpha,R}} 1\,dx\,dy = \iint_{E_{\alpha,r}} 1 \cdot \underbrace{|\det J_\psi|}_{r}\, dr\, d\varphi = \int_0^\alpha d\varphi \int_0^R r\,dr =$$

$$= \int_0^\alpha d\varphi \cdot \frac{1}{2} r^2 \Big|_0^R \quad \frac{1}{2} R^2 \int_0^\alpha d\varphi = \frac{1}{2} R^2 \alpha$$

En particulier, si $\alpha = 2\pi \Rightarrow$ L'aire du disque de rayon R est $A_{2\pi,R} = \frac{1}{2}R^2 \cdot 2\pi = \pi R^2$.

Exercice: Calculer l'aire du disque de rayon 1 en coordonnées cartésiennes



$$D = \{(x,y)\in\mathbb{R}^2 : -1 < x < 1,\ -\sqrt{1-x^2} < y < \sqrt{1-x^2}\}$$

$$\iint_D 1\,dx\,dy = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \int_{-1}^{1} dx \left(\sqrt{1-x^2}+\sqrt{1-x^2}\right) = 2\int_{-1}^{1}\sqrt{1-x^2}\,dx = \dots$$

$x=\sin t$, $x\in[-1,1] \Leftrightarrow t\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 t}\,d\sin t = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\cos t|\cdot\cos t\,dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt =$$

$-\frac{\pi}{2}\le t\le\frac{\pi}{2}$

$\Rightarrow |\cos t| = \cos t \ge 0$

$2\cos^2 t - 1 = \cos 2t$

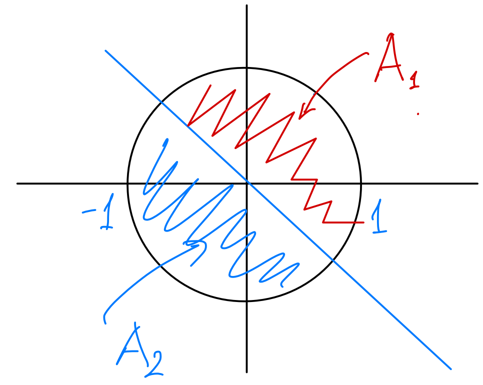
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos 2t + 1)\,dt = \frac{1}{2}\sin 2t + t\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}\Big(\underbrace{\sin\pi}_{=0} - \underbrace{\sin(-\pi)}_{=0}\Big) + \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$\Rightarrow$ Aire (disque de rayon 1) $=\pi$

Ex. $\displaystyle\iint_D (x+y)\,dx\,dy$, $D = \{(x,y)\in\mathbb{R}^2, -1<x<1, -\sqrt{1-x^2}<y<\sqrt{1-x^2}\}$

$= \{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$

$\displaystyle\iint_{A_1}(x+y) = -\iint_{A_2}(x+y)$

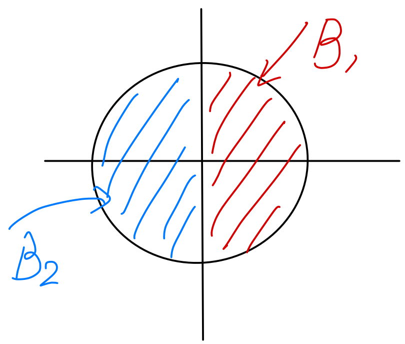


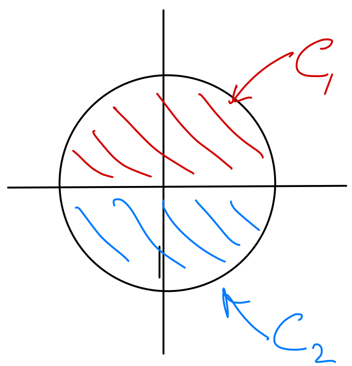
$$\| \qquad \int_0^{2\pi} d\varphi \int_0^1 r(\sin\varphi+\cos\varphi)\,\underbrace{r}_{|J_\varphi|}\,dr =$$

$$= \int_0^{2\pi}(\sin\varphi+\cos\varphi)\,d\varphi \cdot \int_0^1 r^2\,dr = \underbrace{\left(-\cos\varphi+\sin\varphi\Big|_0^{2\pi}\right)}_{=0}\left(\frac{1}{3}r^3\Big|_0^1\right) = 0$$

Autrement, $\displaystyle\iint_D (x+y)\,dx\,dy = \iint_D x\,dx\,dy + \iint_D y\,dx\,dy = 0 + 0 = 0.$

$\displaystyle\iint_{B_1} x = -\iint_{B_2} x$





$\displaystyle\iint_{C_1} y = -\iint_{C_2} y$