Rappel: $\mathbb{R}^n \xrightarrow{\bar{g}} \mathbb{R}^p \xrightarrow{\bar{f}} \mathbb{R}^q$; $\bar{g}$ est dérivable en $\bar{a} \in \mathbb{R}^n$
$\bar{f}$ est dérivable en $\bar{b} = \bar{g}(\bar{a}) \in \mathbb{R}^p$

Alors $\bar{f} \circ \bar{g}$ est dérivable en $\bar{a}$ et $$\boxed{J_{f\circ g}(\bar{a}) = J_{\bar{f}}(\bar{g}(\bar{a})) \cdot J_{\bar{g}}(\bar{a})}$$

matrices de Jacobi

produit matriciel.

Ex. 4 $\mathbb{R}_+ \xrightarrow{\bar{g}} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ $\quad \bar{g}(x) = \begin{pmatrix} x^3 \\ \frac{1}{x} \end{pmatrix}$; $f(u,v) = uv^2$.

$$f\circ\bar{g} = f(g_1(x), g_2(x)) = f\left(x^3; \frac{1}{x}\right) = x^3 \cdot \frac{1}{x^2} = x \Rightarrow f\circ\bar{g}(x) = x \Rightarrow J_{f\circ g}(x) = \boxed{1} = (f\circ\bar{g})'(x).$$

$$J_{\bar{g}} = \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_2}{\partial x} \end{pmatrix} = \boxed{\begin{pmatrix} 3x^2 \\ -\frac{1}{x^2} \end{pmatrix}} \quad ; \quad J_f(u,v) = \nabla f(u,v) = (v^2, 2uv)\Big|_{(\underset{u}{g_1(x)},\, \underset{v}{g_2(x)})} = \boxed{\left(\frac{1}{x^2}, 2x^2\right)}$$

$$J_f(\bar{g}(x)) \cdot J_{\bar{g}}(x) = \boxed{\left(\frac{1}{x^2}, 2x^2\right)} \cdot \boxed{\begin{pmatrix} 3x^2 \\ -\frac{1}{x^2} \end{pmatrix}} = 3 - 2 = \boxed{1} = J_{f\circ\bar{g}}(x).$$

On peut aussi considérer la fonction composée $\bar{g}\circ f : \mathbb{R}_+^2 \xrightarrow{f} \mathbb{R}_+ \xrightarrow{\bar{g}} \mathbb{R}_+^2$

$$\begin{tikzcd} \mathbb{R}_+^2 \arrow[r, "f"] \arrow[rr, bend right, "\bar{g}\circ f"'] & \mathbb{R}_+ \arrow[r, "\bar{g}"] & \mathbb{R}_+^2 \end{tikzcd}$$

$$f(x,y) = xy^2 \; ; \; g(u) = \begin{pmatrix} u^3 \\ \frac{1}{u} \end{pmatrix} \Rightarrow \bar{g}\circ f(x,y) = \bar{g}(xy^2) = \begin{pmatrix} x^3y^6 \\ \frac{1}{xy^2} \end{pmatrix}$$

$$J_{\bar{g}\circ f} = \begin{pmatrix} 3x^2y^6 & 6x^3y^5 \\ -\frac{1}{x^2y^2} & -\frac{2}{xy^3} \end{pmatrix}$$

$$J_f(x,y) = \nabla f(x,y) = \boxed{(y^2, 2xy)}$$

$$J_{\bar{g}} = \begin{pmatrix} 3u^2 \\ -\frac{1}{u^2} \end{pmatrix} \Rightarrow J_{\bar{g}}(\underbrace{f(x,y)}_{xy^2 = u}) = \boxed{\begin{pmatrix} 3x^2y^4 \\ -\frac{1}{x^2y^4} \end{pmatrix}}$$

$$J_{\bar{g}}(\overbrace{xy^2}^{f(x,y)}) \cdot J_f(x,y) = \boxed{\begin{pmatrix} 3x^2y^4 \\ -\frac{1}{x^2y^4} \end{pmatrix}} \cdot \boxed{(y^2, 2xy)} = \boxed{\begin{pmatrix} 3x^2y^6 & 6x^3y^5 \\ -\frac{1}{x^2y^2} & -\frac{2}{xy^3} \end{pmatrix}} = J_{\bar{g}\circ f}(x,y)$$

$$\begin{tikzcd} \mathbb{R}_+^2 \arrow[r, "J_f"] \arrow[rr, bend right, "J_{\bar{g}\circ f}"'] & \mathbb{R}_+ \arrow[r, "J_{\bar{g}}"] & \mathbb{R}_+^2 \end{tikzcd}$$

$\det J_{\bar{g}\circ f} = -6xy^3 + 6xy^3 = 0$, $\operatorname{rang}(J_{\bar{g}\circ f}) = 1$.

(Voir Série 8 pour le cas $\mathbb{R}^2 \xrightarrow{\bar{g}} \mathbb{R}^2 \xrightarrow{\bar{f}} \mathbb{R}^2$).

Application : Changement de variables

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\bar{h}"] \arrow[rr, bend right, "Id"'] & \mathbb{R}^n \arrow[r, "\bar{g}"] & \mathbb{R}^n \end{tikzcd}$$

Si $\bar{h}$ est un changement de variables, et $\bar{g}$ sa fonction réciproque, alors

$$\bar{g} \circ \bar{h}\,(x_1, \dots x_n) = (x_1, \dots x_n)$$

$$\Rightarrow J_{\bar{g}\circ\bar{h}} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix} = Id_{n\times n}$$

Supposons que $\bar{h}$ et $\bar{g}$ sont dérivables sur leurs domaines

$\Rightarrow$ Par le Thm d'une fonction composée $\Rightarrow J_{\bar{g}}(\bar{h}(\bar{a})) \cdot J_{\bar{h}(\bar{a})} = Id_{n\times n}$

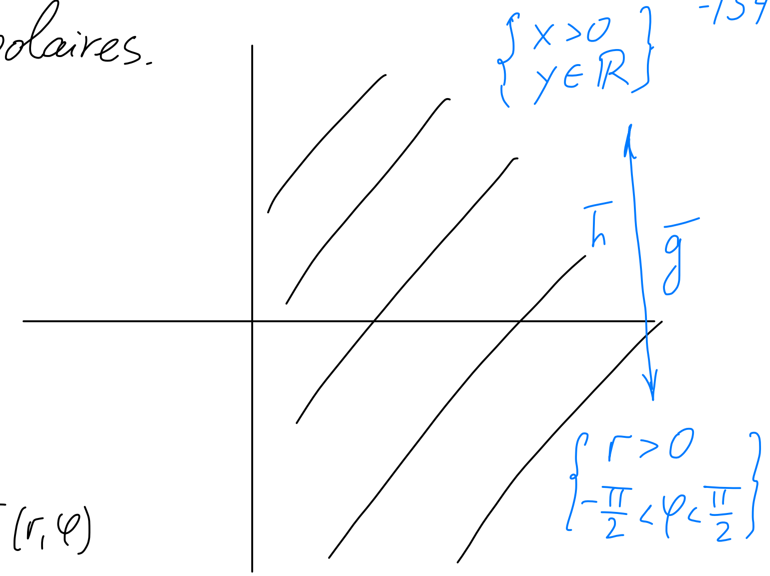
$\Rightarrow J_{\bar{g}}(\bar{h}(\bar{a})) = \left(J_{\bar{h}(\bar{a})}\right)^{-1}$ et on a $(\det J_{\bar{g}}) \cdot (\det J_{\bar{h}}) = 1$

Remarque. Soit $\bar{g}$ continûment dérivable, $\bar{g} : E^{\subset \mathbb{R}^n} \to D^{\subset \mathbb{R}^n}$

Alors $\bar{g}$ est bijective dans un voisinage de $\bar{a} \in E \iff (\det J_{\bar{g}})(\bar{a}) \neq 0$.

Ex5. Changement de variable en coordonnées polaires.

$\bar h(x,y) = (r,\varphi)$; $\bar g(r,\varphi) = (x,y)$



$$\bar g(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi\end{pmatrix} \quad ; \quad h(x,y) = \begin{pmatrix}\sqrt{x^2+y^2} \\ \mathrm{Arctg}\,\frac{y}{x}\end{pmatrix}$$

$$(\bar h\circ\bar g)(r,\varphi) = (r,\varphi) \Rightarrow J_{\bar h\circ \bar g} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = J_{\bar h(\bar g(x,y))}\cdot J_{\bar g(r,\varphi)}$$

$$J_{\bar g}(r,\varphi) = \begin{pmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{pmatrix}$$

$$J_{\bar h} = (J_{\bar g})^{-1} = \frac1r \begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -\sin\varphi & \cos\varphi\end{pmatrix} = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\frac{\sin\varphi}{r} & \frac{\cos\varphi}{r}\end{pmatrix}$$

$$\det J_{\bar g} = r\cos^2\varphi + r\sin^2\varphi = r > 0$$
$$r \neq 0$$

$$\cos\varphi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \quad ; \quad \sin\varphi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow J_{\bar h}(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y}\end{pmatrix} = \begin{pmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} & \frac{\frac{1}{x}}{1+\frac{x^2}{y^2}}\end{pmatrix}$$

Exercice : vérifier directement

§ 4.6. Dérivée d'une intégrale qui dépend d'un paramètre.

Thm. Soit $f : [a,b] \times \underset{\text{ouvert}}{I} \overset{\subset \mathbb{R}}{} \longrightarrow \mathbb{R}$ telle que $\frac{\partial f}{\partial y}$ est continue sur $[a,b]\times I$.

Alors $g(y) = \int_a^b f(x,y)\,dx$ est de classe C^1 sur I et on a :

$$g'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx \quad \text{pour tout } y \in I.$$

Dém:

$$\frac{g(y)-g(y_0)}{y-y_0} = \frac{1}{y-y_0}\int_a^b \big(f(x,y)-f(x,y_0)\big)\,dx = \frac{1}{\cancel{y-y_0}}\int \underbrace{\frac{\partial f}{\partial y}(x,\overset{\text{entre } y \text{ et } y_0}{\tilde{y}})\cancel{(y-y_0)}}_{\text{Thm des accroissements finis par rapport à } y}\,dx =$$

$$= \int_a^b \frac{\partial f}{\partial y}(x,\tilde{y})\,dx \xrightarrow[\substack{y\to y_0\\ \tilde{y}\to y_0}]{\frac{\partial f}{\partial y}\text{ continue}} \int_a^b \frac{\partial f}{\partial y}(x,y_0)\,dx = g'(y_0)$$

$\boxed{///}$

Ex 1. $g(y) = \int_0^{\pi/2} \sin(xy)\,dx$; calculer $g'(y)$ $y \in \mathbb{R}$

$$g'(y) \overset{\text{Thm}}{=} \int_0^{\pi/2} \cos(xy)\cdot x\,dx \overset{y\neq 0}{=} \frac{1}{y}\int_0^{\pi/2} \underbrace{x}_{u}\, d\big(\underbrace{\sin(xy)}_{v}\big) = \frac{x}{y}\sin(xy)\Big|_0^{\frac{\pi}{2}} - \frac{1}{y}\int_0^{\frac{\pi}{2}}\sin(xy)\,dx = \cdots \longrightarrow$$

$$\frac{1}{y^2}\cos(xy)\Big|_0^{\frac{\pi}{2}}$$

$\underline{y=0}$: $$\int_0^{\frac{\pi}{2}} x\,dx = \frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}.$$

$$\frac{\pi}{2y}\sin\left(\frac{\pi y}{2}\right)+\frac{1}{y^2}\cos(xy)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2y}\sin\left(\frac{\pi y}{2}\right)+\frac{1}{y^2}\left(\cos\left(\frac{\pi y}{2}\right)-1\right) = g'(y)$$

<u>Directement</u> $g(y) = \int_0^{\frac{\pi}{2}} \sin(xy)\,dx \overset{y\neq 0}{=} -\frac{1}{y}\cos(xy)\Big|_0^{\pi/2} = -\frac{1}{y}\left(\cos\left(\frac{\pi y}{2}\right)-1\right) =$

$$g'(y) = -\frac{1}{y^2}+\frac{1}{y^2}\cos\left(\frac{\pi y}{2}\right)+\frac{1}{y}\sin\left(\frac{\pi y}{2}\right)\cdot\frac{\pi}{2}$$

<u>Exercice:</u> Calculer la limite $\lim_{y\to 0} g(y)$ et $g'(0)$; aussi $\lim_{y\to 0} g'(y)$

Astuce: Utiliser le DL pour $\cos t$ autour de $t=0$: $\cos t = 1-\frac{1}{2}t^2+\frac{1}{4!}t^4-\dots$

$$\cos\left(\frac{\pi}{2}y\right) = 1-\frac{1}{2}\left(\frac{\pi}{2}y\right)^2+\frac{1}{4!}\left(\frac{\pi}{2}y\right)^4-\dots \quad \text{lorsque } y\to 0.$$

$$\lim_{y\to 0} g(y) = \lim_{y\to 0}\frac{1-\cos\left(\frac{\pi y}{2}\right)}{y} = \lim_{y\to 0}\frac{\cancel{1}-\left(\cancel{1}-\frac{1}{2}\left(\frac{\pi}{2}y\right)^2+\dots y^2\varepsilon(y)\dots\right)}{y} = \underline{0} \equiv g(0)$$

$$g'(0) = \lim_{y\to 0}\frac{g(y)-g(0)}{y} = \lim_{y\to 0}\frac{1-\cos\left(\frac{\pi y}{2}\right)-0}{y^2} = \lim_{y\to 0}\frac{\cancel{1}-\left(\cancel{1}-\frac{1}{2}\left(\frac{\pi}{2}y\right)^2+y^2\varepsilon(y)\right)}{y^2} = \underline{\frac{\pi^2}{8}} \quad \text{comme avant.}$$

$$\lim_{y\to 0} g'(y) = \lim_{y\to 0}\left(\frac{\pi}{2}\frac{\sin\left(\frac{\pi y}{2}\right)}{y}+\frac{\cos\left(\frac{\pi}{2}y\right)-1}{y^2}\right) = \lim_{y\to 0}\left(\frac{\pi}{2}\right)^2\underbrace{\frac{\sin\left(\frac{\pi y}{2}\right)}{\frac{\pi y}{2}}}_{\xrightarrow{y\to 0} 1}+\lim_{y\to 0}\frac{\cancel{1}-\frac{1}{2}\left(\frac{\pi}{2}y\right)^2+y^2\varepsilon(y)-\cancel{1}}{y^2} =$$

$$= \left(\frac{\pi}{2}\right)^2-\frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \underline{\frac{\pi^2}{8}} = g'(0). \Rightarrow \text{La dérivée } g'(y) \text{ est continue en } y=0.$$

Rappel: Analyse I Thm fondamental du calcul intégral:

$$\frac{d}{dt}\left(\int_a^t f(y)dy\right) = f(t) \quad , \quad \frac{d}{dt}\left(\int_t^b f(y)dy\right) = \frac{d}{dt}\left(-\int_b^t f(y)dy\right) = -f(t).$$

f continue

On peut combiner les resultats:

Thm. Soient $g, h : I \xrightarrow{C^1} \mathbb{R}$ fonctions continûment dérivables, et $f : \overset{x}{J} \times \overset{t}{I} \to \mathbb{R}$ telle que $\frac{\partial f}{\partial t}$ est continue sur I.

ouverts

Alors $F(t) = \int_{h(t)}^{g(t)} f(x,t)\,dx$ est continûment dérivable sur I, et on a:

$$\boxed{F'(t) = f(g(t),t)\cdot g'(t) - f(h(t),t)\cdot h'(t) + \int_{h(t)}^{g(t)} \frac{\partial f}{\partial t}(x,t)\,dx}$$

Dém: $F : \underset{g,h,t}{\mathbb{R}^3} \to \mathbb{R}$; $F = \int_h^g f(x,t)\,dx$; $(g,h,t) : \underset{t}{\mathbb{R}} \longrightarrow \underset{(g(t),h(t),t)}{\mathbb{R}^3}$ $\Rightarrow$

$$\mathbb{R} \xrightarrow{(g,h,t)} \mathbb{R}^3 \xrightarrow{F} \mathbb{R}$$

Alors, par le Thm de la derivée d'une fonction composée, on a :

$$F'(t) = J_{F\circ(g,h,t)(t)} = \nabla F(g,h,t) \cdot \begin{pmatrix} g'(t) \\ h'(t) \\ 1 \end{pmatrix} =$$

$$= \underbrace{\frac{\partial F}{\partial g}}_{\text{Thm. fond.}\ = f(g(t),t)} \cdot g'(t) + \frac{\partial F}{\partial h} \cdot h'(t) + \frac{\partial F}{\partial t} \cdot 1 = f(g(t),t) \cdot g'(t) - f(h(t),t) \cdot h'(t) + \underbrace{\int_{h(t)}^{g(t)} \frac{\partial}{\partial t} f(x,t)\,dx}_{\text{Thm sur la dérivation d'une intégrale}}$$

$\blacksquare$

Ex 2. $F(t) = \int_{2t}^{3t^2} e^{t+x}\,dx$: $F'(t) = ?$

On a : $h(t) = 2t$, $g(t) = 3t^2$, $f(x,t) = e^{x+t}$

Par le Thm : $F'(t) = e^{3t^2+t} \cdot g'(t) - e^{2t+t} \cdot h'(t) + \int_{2t}^{3t^2} \frac{\partial e^{x+t}}{\partial t}\,dx =$

$$= 6t \cdot e^{3t^2+t} - 2e^{3t} + \int_{2t}^{3t^2} e^x e^t\,dx = 6t\,e^{3t^2+t} - 2e^{3t} + e^t\left(e^{3t^2} - e^{2t}\right) =$$

$$= (6t+1)e^{3t^2+t} - 3e^{3t}.$$

Directement: $F(t) = \int_{2t}^{3t^2} e^t e^x \, dx = e^t\left(e^{3t^2} - e^{2t}\right) = e^{3t^2+t} - e^{3t}$.

$F'(t) = (6t+1)e^{3t^2+t} - 3e^{3t}$, le même resultat.