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Équations différentielles

(1) $xyy' = 1 + x^2$ EDVS : $\int y\,dy = \int \frac{1+x^2}{x}\,dx \rightarrow \frac{1}{2}y^2 = \ln|x| + \frac{1}{2}x^2 + C$

$y^2 = \ln x^2 + x^2 + C$

$\rightsquigarrow$ Conditions initiales $\Rightarrow C \Rightarrow y(x) = \ldots$

(2) $y' + \frac{2y}{x} = \underbrace{x^2}_{f(x)}$, $p(x) = \frac{2}{x}$ EDL1 ; $P(x) = \int p(x)\,dx = \int \frac{2\,dx}{x} = 2\ln|x| = \ln x^2$

$y_{hom}(x) = C e^{-P(x)} = \frac{C}{x^2}$ vérifier !

$c(x) = \int f(x) e^{P(x)} = \int x^2 \cdot e^{\ln x^2} dx = \int x^4 dx =$

$= \frac{1}{5}x^5 \Rightarrow y = \frac{1}{5}x^3 + \frac{C}{x^2}$ vérifier

$y_p = c(x) \cdot e^{-P(x)} = \frac{1}{5}x^5 \cdot e^{-\ln x^2} = \frac{1}{5}x^3$ $\Rightarrow$ Cond. initiales $\Rightarrow C \Rightarrow y(x) = \ldots$

(3) $y' = -\frac{3e^x}{1-e^x}\tan y \cdot \cos^2 y$ EDVS : $\int \frac{dy}{\tan y \cos^2 y} = -\int \frac{3e^x\,dx}{1-e^x}$

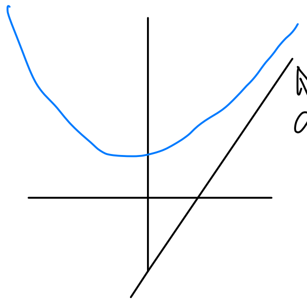
$\int \frac{d(\tan y)}{\tan y} = +\int \frac{3\,d(1-e^x)}{1-e^x} \Rightarrow \ldots$

Equations différentielles

(4) $y'' + 2y' + y = 2x+1$ EDL2 à coeff. constants $\lambda^2+2\lambda+1 = (\lambda+1)^2 \Rightarrow \lambda = -1$

$\Rightarrow y_{hom} = C_1 e^{-x} + C_2 x e^{-x}$

Ansatz : $Ax + B = y_p \rightarrow y' = A,\ y'' = 0$



$\Rightarrow 2A + Ax + B = 2x+1$ $y_p = 2x-3 \rightarrow$ vérifier

$A = 2,\ 4 + B = 1 \Rightarrow B = -3$

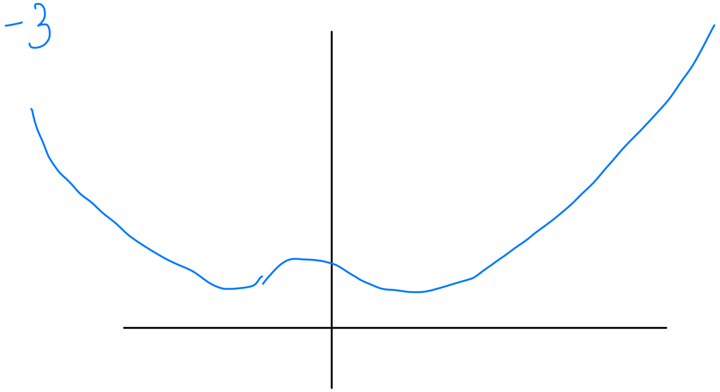
$\Rightarrow y(x) = 2x - 3 + C_1 e^{-x} + C_2 x e^{-x}$ $\leadsto$ cond. initiales $\Rightarrow C_1, C_2$

(5) $y'' - y' - 2y = 20 \sin(2x)$ EDL2 à coeff const $(\lambda-2)(\lambda+1) \Rightarrow \lambda = 2, \lambda = -1$

$y_{hom} = C_1 e^{2x} + C_2 e^{-x}$

Ansatz : $A \sin 2x + B \cos 2x \leadsto B = 1,\ A = -3$

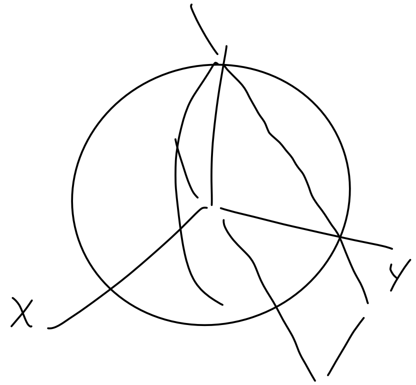
$y(x) = -3 \sin 2x + \cos 2x + C_1 e^{2x} + C_2 e^{-x}$

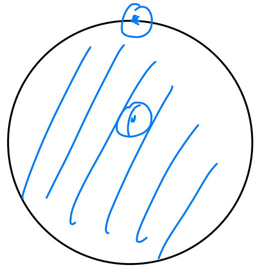


Topologie dans $\mathbb{R}^n$

Fermé, ouvert, ni fermé ni ouvert ?
Borné, non borné ?

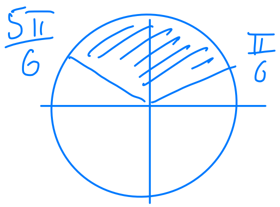
(1) $E = \{(x,y,z) \in \mathbb{R}^3 : x = y,\ x^2 + y^2 + z^2 \underline{<} 20\}$





ni ouvert ni fermé
borné

(2) $D = \{(x,y) \in \mathbb{R}^2 : \sin(x + ty) > \frac{1}{2}\}$, $t \in \mathbb{R}$ fixé

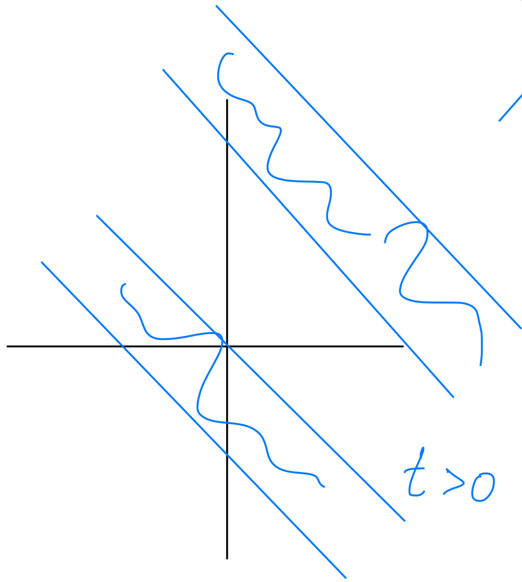


$$\frac{\pi}{6} + 2k\pi < x + ty < \frac{5\pi}{6} + 2k\pi$$

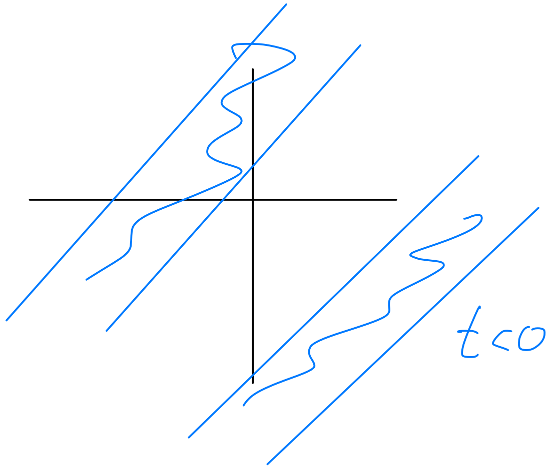
$$-ty + \frac{\pi}{6} + 2k\pi < x < -ty + \frac{5\pi}{6} + 2k\pi$$

$k \in \mathbb{Z}$

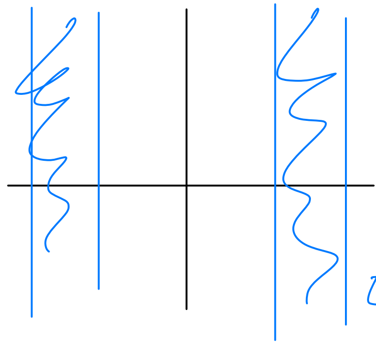
ouvert
non borné



$t > 0$

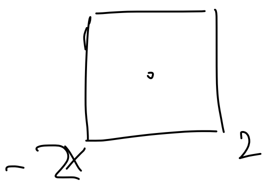


$t < 0$



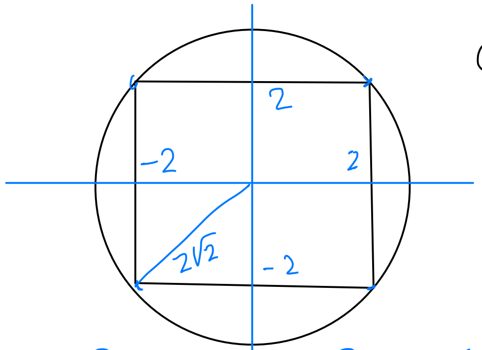
$t = 0$

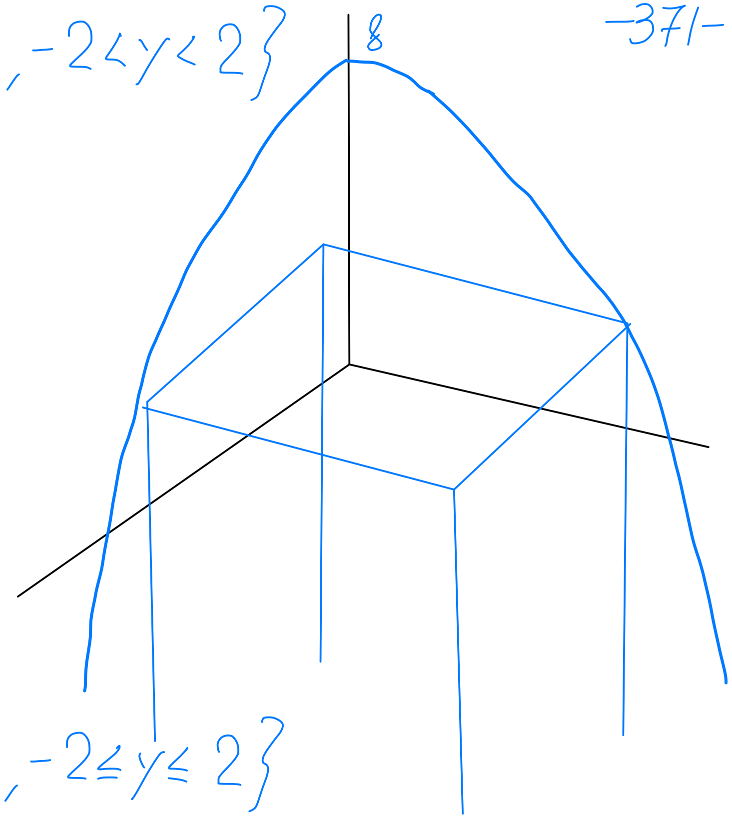
(3) $E_1 = \{(x,y,z)\in\mathbb{R}^3 : z \le 8-x^2-y^2\} \cap \{z<0, -2<x<2, -2<y<2\}$



Si $z=0 \Rightarrow x^2+y^2=8$

cercle de rayon $2\sqrt{2}$





$\Rightarrow E_1 = \{z<0, -2<x<2, -2<y<2\}$ ouvert

non borné

(4) $E_2 = \{(x,y,z)\in\mathbb{R}^3 : z < 8-x^2-y^2\} \cap \{z\le 0, -2\le x\le 2, -2\le y\le 2\}$

$(2,2,0)$: $0 \overset{?}{<} 8-4-4$ non $\Rightarrow (2,2,0) \notin E_2$

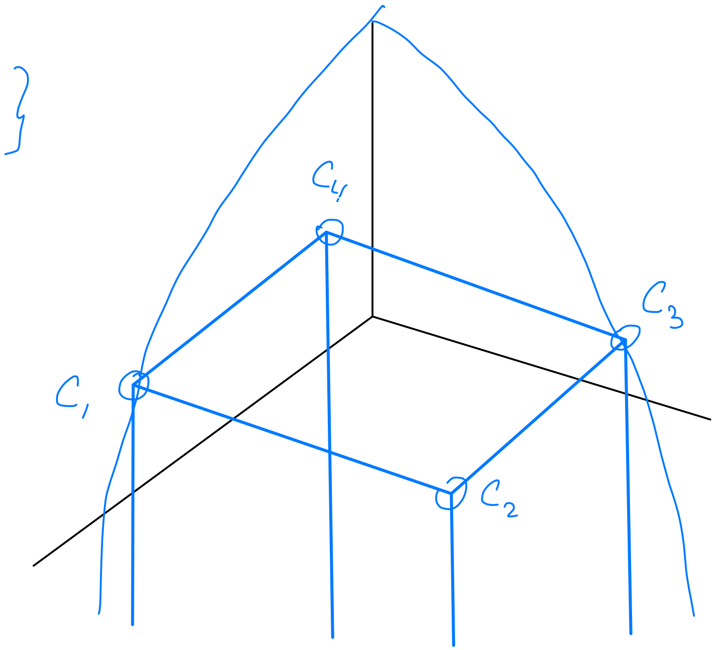
$E_2 = \{z\le 0, -2\le x\le 2, -2\le y\le 2\} \setminus \{C_1, C_2, C_3, C_4\}$

$\overset{C_1}{(2,2,0)}$ $\overset{C_2}{(2,-2,0)}$ $\overset{C_3}{(-2,2,0)}$ $\overset{C_4}{(-2,-2,0)}$

cylindre carré sans les coins.

ni ouvert ni fermé

non borné.



Limites

(a) $\bar{a}_n = \left(\frac{n^2+3n}{n^2-1},\ n\left(e^{\frac{(-1)^n}{n}}-1\right),\ n\left(\left(1+\frac{1}{n}\right)^{-1}-1\right)\right)$

$n \to \infty$: $\frac{n^2+3n}{n^2-1} \to 1$, $\quad n\left(e^{\frac{(-1)^n}{n}}-1\right)$ n'existe pas, $\quad n\left(\left(1+\frac{1}{n}\right)^{-1}-1\right) \to -1$

$\lim_{n\to\infty} \bar{a}_n$ n'existe pas

$$n\left(e^{\frac{(-1)^n}{n}}-1\right) \sim \left(n\left(\cancel{1}+\frac{(-1)^n}{n}+\frac{1}{2n^2}+\dots-\cancel{1}\right)\right) \to (-1)^n$$

$$n\left(\left(1+\frac{1}{n}\right)^{-1}-1\right) = n\left(\frac{1}{1+\frac{1}{n}}-1\right) =$$

$$n\left(\cancel{1}-\frac{1}{n}+\frac{1}{n^2}-\dots-\cancel{1}\right) \to -1$$

$$(1+x)^\alpha \sim 1+\alpha x+\frac{\alpha(\alpha-1)x^2}{2}+\dots\frac{\alpha(\alpha-1)\dots(\alpha-k)}{k!}x^k+\dots$$

x petit

(b) $\lim_{(x,y)\to(0,0)} \frac{e^{x^3}-1}{x^2+y^2} = 0$

$$\underset{\downarrow 0}{0} < \underset{\downarrow 0}{\left|\frac{e^{x^3}-1}{x^2+y^2}\right|} \le \left|\frac{e^{x^3}-1}{x^2}\right| \sim \underset{\downarrow 0}{\left|\frac{\cancel{1}+x^3+\frac{x^6}{2}+\dots-\cancel{1}}{x^2}\right|} \quad |x|\to 0$$

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2\cos^2(xy)}{x^2+y^{10}} = 0$

$$\underset{\downarrow 0}{0} < \underset{\downarrow 0}{\left|\frac{x^2y^2\overbrace{\cos^2(xy)}^{<1}}{x^2+y^{10}}\right|} < \frac{|x^2y^2|}{|x^2+y^{10}|} < \left|\frac{x^2y^2}{x^2}\right| = \underset{\downarrow 0}{y^2}$$

$\sqrt{x^2+y^2} \to 0 \Rightarrow |y| \to 0$

$$|y| = \sqrt{y^2} < \sqrt{x^2+y^2} \longrightarrow 0$$

Dérivabilité

Soit $f : \underset{\text{ouvert}}{E} \xrightarrow{\;\subset \mathbb{R}^n\;} \mathbb{R}$, $\bar{a} \in E$.

$$\underset{\text{sur } E}{\text{Classe } C^2} \underset{\text{Ex4}}{\overset{\text{déf}}{\underset{\not\Leftarrow}{\Longrightarrow}}} \underset{\text{sur } E}{\text{Classe } C^1} \underset{\text{Ex5}}{\overset{\text{Thm 2}}{\underset{\not\Leftarrow}{\Longrightarrow}}} \underset{\text{en } \bar{a} \in E}{\text{Dérivable}} \underset{\text{Ex2}}{\overset{\text{Thm 1}}{\underset{\not\Leftarrow}{\Longrightarrow}}} \underset{\forall \bar{v} \neq \bar{0}}{Df(\bar{a}, \bar{v}) \text{ existent,}} \underset{\text{Ex1}}{\overset{\text{déf}}{\underset{\not\Leftarrow}{\Longrightarrow}}} \frac{\partial f}{\partial x_k}(\bar{a}) \text{ existent } \forall k = 1 \ldots n$$

$\nabla f(\bar{a})$ existe

$$\text{Dérivable en } \bar{a} \in E \;\overset{\text{Thm 1}}{\Longrightarrow}\; \text{Continue en } \bar{a} \in E \qquad \text{Continue en } \bar{a} \in E \;\overset{\text{Ex6}}{\not\Longrightarrow}\; \text{Dérivable en } \bar{a} \in E$$

$$Df(\bar{a}, \bar{v}) \text{ existent, } \forall \bar{v} \neq \bar{0} \;\overset{\text{Ex2}}{\not\Longrightarrow}\; \text{Continue en } \bar{a} \in E$$

$$\frac{\partial f}{\partial x_k}(\bar{a}) \text{ existent } \forall k = 1 \ldots n,\ \nabla f(\bar{a}) \text{ existe} \;\overset{\text{Ex1}}{\not\Longrightarrow}\; \text{Continue en } \bar{a} \in E$$

Dérivabilité

(1) $$f(x,y) = \begin{cases} \dfrac{y \sin^4 x}{x^4+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Dérivées partielles existent en $(0,0)$? Dérivable en $(0,0)$? C^1 sur $\mathbb{R}^2$?

(1) $$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{\frac{0\cdot \sin^4 t}{t^4+0} - 0}{t} = \lim_{t\to 0} \frac{0}{t^5} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{\frac{t\cdot \sin^4 0}{0+t^4} - 0}{t} = \lim_{t\to 0} \frac{0}{t^5} = 0$$

$\Rightarrow \nabla f(0,0) = (0,0)$

(2) Dérivabilité : $r(x,y) = f(x,y) - f(0,0) - \underbrace{\langle \nabla f(0,0), (x,y)\rangle}_{=0} = f(x,y)$

$$\lim_{(x,y)\to(0,0)} \frac{r(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{y \sin^4 x}{(x^4+y^4)\sqrt{x^2+y^2}}$$ n'existe pas $\Rightarrow$ f n'est pas dérivable en $(0,0)$.

$\left(\frac{1}{k}, \frac{1}{k}\right)$: $$\lim_{k\to\infty} \frac{\frac{1}{k} \sin^4\left(\frac{1}{k}\right)}{\frac{2}{k^4}\sqrt{2}\,\frac{1}{k}} = \lim \left(\frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}}\right)^4 \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(t,t)

$\left(\frac{1}{k}, 0\right)$: $$\lim_{k\to\infty} \frac{0}{\frac{1}{k^4}\cdot\frac{1}{k}} = 0$$

Théorème de la fonction implicite

$(0, \sqrt[4]{2}, \sqrt[3]{3}) \Rightarrow x = g(y,z)$?

(1) $F(x,y,z) = x^2 + y^4 + z^6 - 11 = 0$ autour de $(3,-1,1) \Rightarrow x = f(y,z)$?

Si oui, trouver $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial y}$

$$\frac{\partial F}{\partial x}(x,y,z) = 2x \Big|_{(3,-1,1)} = 6 \neq 0$$

$$\Rightarrow \exists\ x = f(x,y) \text{ et } \frac{\partial f}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} = -\frac{4y^3}{2x}\Big|_{(3,-1,1)} = -\frac{-4}{6} = \underline{\frac{2}{3}}$$

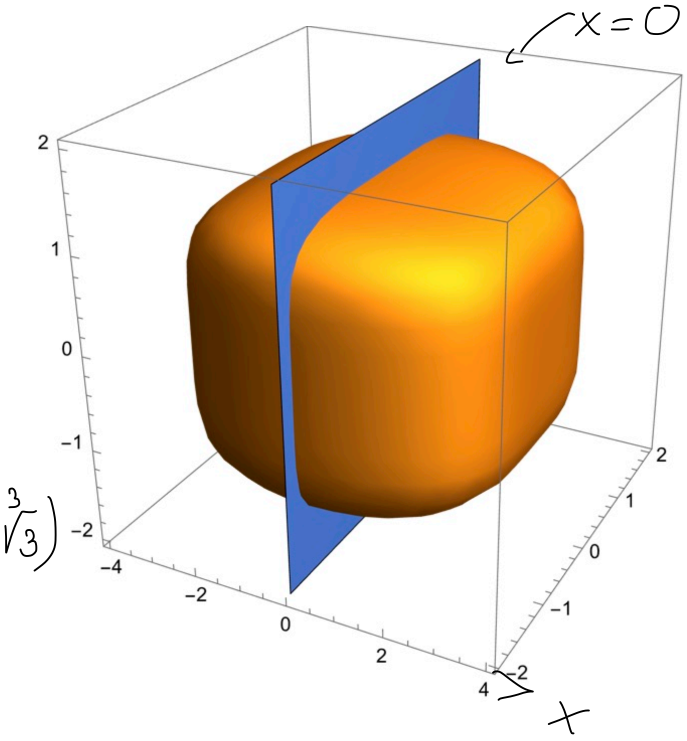
$$\frac{\partial F}{\partial x}(x,y,z) = 2x \Big|_{(0, \sqrt[4]{2}, \sqrt[3]{3})} = 0$$

$$x^2 = 11 - y^4 - z^6$$

$$x = \pm\sqrt{11 - y^4 - z^6}$$

2 valeur en tout voisinage de $(0, \sqrt[4]{2}, \sqrt[3]{3})$

$\Rightarrow$ $x = g(y,z)$ n'existe pas



Polynômes de Taylor

$$P_2 f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)\cdot(y-b)^2\right)$$

(1) $f(x,y) = \cos(2x+xy)$ d'ordre 2 autour de $(x,y)=(1,0)$

$$\cos\left(2(x-1)+2+(x-1)y+y\right) = \cos\big(2+\underbrace{y+2(x-1)+y(x-1)}_{u \text{ petit}}\big) = \cos 2\cos u - \sin 2 \sin u \overset{\text{ordre } 2}{\approx}$$

$$\cos 2\left(1-\tfrac{1}{2}u^2\right) - \sin 2\cdot u = \cos 2\left(1-\tfrac{1}{2}\left(y+2(x-1)+y(x-1)\right)^2\right) - \sin 2\cdot\left(y+2(x-1)+y(x-1)\right) \underset{\text{en } y,\,(x-1)}{\overset{\text{ordre } 2}{\simeq}}$$

$$\cos 2\left(1-\tfrac{1}{2}\left(y^2+4(x-1)^2+4y(x-1)\right)\right) - \sin 2\left(y+2(x-1)+y(x-1)\right) =$$

$$= \cos 2 - (\sin 2)y - 2(\sin 2)(x-1) - \tfrac{1}{2}(\cos 2)y^2 - 2(\cos 2)(x-1)^2 - \left(2(\cos 2)+(\sin 2)\right)y(x-1)$$

---

(2) $f(x,y,z) = \ln(1+\underbrace{x^2+z}_{t})e^{\overbrace{y+x}^{s}}$ d'ordre 2 autour de $(x,y,z)=(0,0,0)$

$$\ln(1+t) = t-\tfrac{1}{2}t^2+\ldots \qquad e^s = 1+s+\tfrac{1}{2}s^2+\ldots$$

$$f(x,y,z) \overset{\text{ordre } 2}{\approx} \left((x^2+z)-\tfrac{1}{2}(x^2+z)^2\right)\left(1+(x+y)+\tfrac{1}{2}(x+y)^2\right) \overset{\text{ordre } 2}{\simeq} (x^2+z)-\tfrac{1}{2}z^2+z(x+y)$$

$$P_2 f(x,y,z) = \underline{z+x^2+xz+zy}$$

Matrice Jacobienne

$$\Rightarrow \quad J_{\bar f\circ\bar g}(\bar a) = J_{\bar f}(\bar g(\bar a))\cdot J_{\bar g}(\bar a)$$

(1) $g: \mathbb{R}^2 \to \mathbb{R}^3$, $\bar g(x,y) = \begin{pmatrix} x^2 \\ xy \\ 2y \end{pmatrix}$; $f: \mathbb{R}^3 \to \mathbb{R}$, $\nabla f(1,\pi,2\pi) = (0,0,1)$

Trouver $\dfrac{\partial (f\circ\bar g)}{\partial y}$ en $(x,y) = (1,\pi)$.

$$\begin{tikzcd} \underset{\binom{x}{y}}{\mathbb{R}^2} \arrow[r, "\bar g"] \arrow[rr, bend right, "f\circ\bar g"'] & \underset{\begin{pmatrix} u\\ v\\ w\end{pmatrix}}{\mathbb{R}^3} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$J_{\bar g}(x,y) = \begin{pmatrix} 2x & 0 \\ y & x \\ 0 & 2 \end{pmatrix}\Bigg|_{(1,\pi)} = \begin{pmatrix} 2 & 0 \\ \pi & 1 \\ 0 & 2 \end{pmatrix}$$

$$g(1,\pi) = \begin{pmatrix} 1 \\ \pi \\ 2\pi \end{pmatrix}$$

$$\Rightarrow J_{f\circ\bar g}(1,\pi) = \nabla f(1,\pi,2\pi)\cdot J_{\bar g}(1,\pi) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ \pi & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f\circ\bar g}{\partial x} & \dfrac{\partial f\circ\bar g}{\partial y} \end{pmatrix}\Bigg|_{(1,\pi)}$$

$$\frac{\partial (f\circ\bar g)}{\partial y}(1,\pi) = \underline{2}$$

Plan tangent

(1) Soit $f: \mathbb{R}^2 \to \mathbb{R}$ de classe C^1 telle que $f(1,0)=6$ $\quad z = 3+3x-2y$

avec le polynôme de Taylor d'ordre 1 autour de $(1,0)$ $\quad P_1(x,y) = 3+3x-2y$

Trouver l'équation du plan tangent au graphique $z = f(x,y)$ à $(x,y,z) = (1,0,6)$

$F = z - f(x,y) \Rightarrow \langle \nabla F(x,y,z)\big|_{(1,0,6)}, (x-1, y-0, z-6) \rangle = 0$ l'équation du plan tangent

$$\left(-\frac{\partial f}{\partial x} \quad -\frac{\partial f}{\partial y} \quad 1\right) = \,?$$

$$P_1(x,y) \Rightarrow 3 + \underbrace{3}_{\frac{\partial f}{\partial x}(1,0)}(x-1) + 3 \underbrace{-2}_{\frac{\partial f}{\partial y}(1,0)}y = 6 + 3(x-1) - 2y \quad = f(1,0) + \frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)(y-0)$$

$$\Rightarrow \nabla F(1,0,6) = (-3, 2, 1)$$

$$\langle (-3, 2, 1)(x-1, y-0, z-6) \rangle = -3(x-1) + 2y + z - 6 = 0$$

$$z - 3x - 3 + 2y = 0 \;; \quad z = 3 + 3x - 2y$$

$$P_1(x,y) = 3 + 3x - 2y$$

Plus généralement, si $f : \mathbb{R}^2 \to \mathbb{R}$ de classe C^2, telle que $f(a,b) = c$ et son polynôme de Taylor d'ordre 2 autour de (a,b) est donné par $P_2 f(x,y) = c + d_1(x-a) + d_2(y-b)$, alors l'équation du plan tangent au graphique $z = f(x,y)$ au point (a,b,c) est donnée par $z = c + d_1(x-a) + d_2(y-b)$.

Dém: $\left\langle \left(-\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1\right), (x-a, y-b, z-c)\right\rangle = 0$ est l'équation du plan tangent.

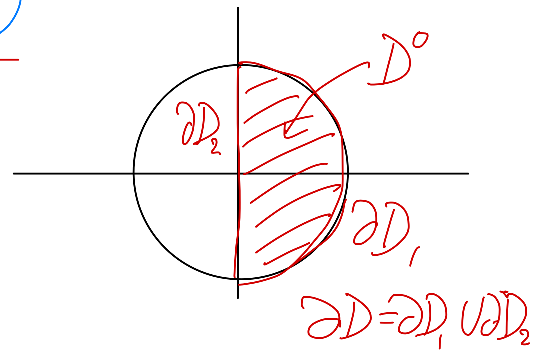
$$\Rightarrow -\frac{\partial f}{\partial x}(a,b)\cdot(x-a) - \frac{\partial f}{\partial y}(a,b)\cdot(y-b) + z - c = 0$$

$$\Rightarrow z = c + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

ce qui est exactement le polynôme de Taylor de f d'ordre 1 autour de (a,b).

(par définition du polynôme de Taylor) ▨

Extremum d'une fonction (sous contrainte)



(1) $f(x,y) = y + 2x \quad , \quad D = \{x^2+y^2 \leq 1 \text{ et } x \geq 0\}$

(1) Extremum sur le $\frac{1}{2}$ disque ouvert D° : $\{x^2+y^2<1,\ x>0\}$

$\nabla f(x,y) = (2,1) \neq (0,0) \Rightarrow$ pas de point stationnaires dans D°.

(2) Extremum sur $\partial D_1 \Rightarrow f$ sous contrainte $g(x,y) = x^2+y^2-1 = 0$.

$\nabla g(x,y) = (2x, 2y) \neq (0,0) \quad x>0$ si $x^2+y^2=1$

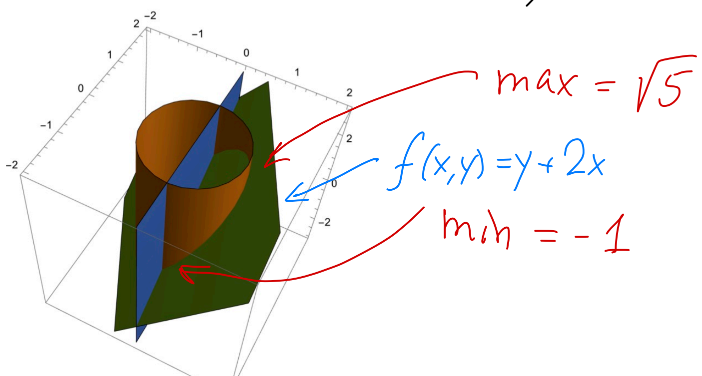
$\Rightarrow$ Thm de Lagrange $\exists \lambda \in \mathbb{R} : \nabla f(x,y) = \lambda \nabla g(x,y) \quad \begin{cases} 2 = \lambda \cdot 2x \\ 1 = \lambda 2y \end{cases} \Rightarrow \lambda = \frac{1}{x}$, $x>0$

$\Rightarrow 1 = \frac{2y}{x} \Rightarrow x = 2y \rightarrow 4y^2+y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{5}} \Rightarrow x = \pm\frac{2}{\sqrt{5}} \quad x>0$

$\Rightarrow \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \qquad f\left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \boxed{\sqrt{5}}$ max sur D

Extremum sur $x=0$, $-1 \leq y \leq 1 \Rightarrow f(x,y) = y \Rightarrow \min_{\partial D_2} f = \boxed{-1}$, $(0,-1)$ min sur D

$\max_{\partial D_2} f = 1 \quad (0,1)$



Examen final: 16 juin, 9:15 - 12:45

Salles CE1515, CE14, CE15, CE16, PO01

(voir les informations sur Moodle)

Comment réussir son examen.

(1) Lire et mémoriser les résumés

(2-3) Relire les notes du cours

+ questions supplémentaires

Faire séries supplémentaires, examens 2015, 2016, 2024

(4) Relire les séries d'exercices. + les demonstrations + examen 2018 corrigés detaillés.

(5) Faire examens 2017, 2019, 2020, 2021, 2023, parties specifiques 2018, 2022, 2023, 2024

(6) Relire les résumés !!

(7) Faire examen - 2022

Bon courage !

et bonnes vacances !!