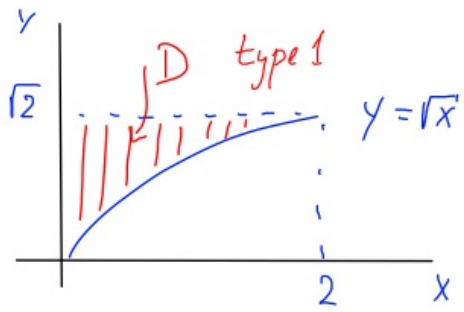
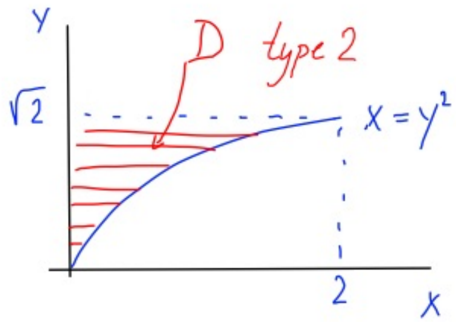
Rappel : Méthodes d'intégration des fonction de plusieurs variables.

$\boxed{1}$ Théorème de Fubini $\Rightarrow$ Changement d'ordre d'intégration.

Ex



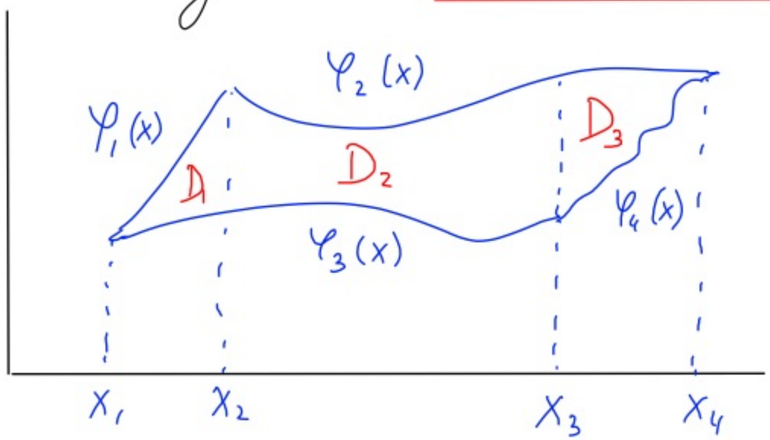


$$\int_0^2 dx \int_{\sqrt{x}}^{\sqrt{2}} f(x,y)\,dy = \int_0^{\sqrt{2}} dy \int_0^{y^2} f(x,y)\,dx$$

$f : \bar{D} \to \mathbb{R}$ continue

$\boxed{2}$ L'additivité de l'intégrale $\Rightarrow$ Division du domaine d'intégration.

$$D = \underbrace{\bigcup_{i=1}^{3} D_i}_{\text{domaines réguliers}}$$



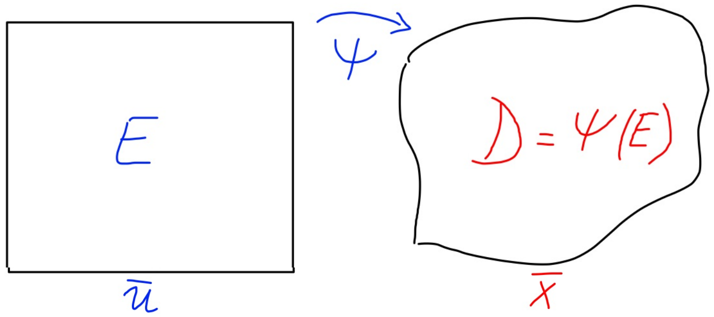
Il faut diviser le domaine en réunion des domaines réguliers et utiliser l'additivité de l'intégrale.

$$\iint_D f(x,y)\,dx\,dy = \int_{x_1}^{x_2} \left(\int_{\varphi_3(x)}^{\varphi_1(x)} f(x,y)\,dy \right) dx + \int_{x_2}^{x_3} \left(\int_{\varphi_3(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx + \int_{x_3}^{x_4} \left(\int_{\varphi_4(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx.$$

$\boxed{3}$ Changement de variables.

Thm Soit $E \subset \mathbb{R}^n$ un sous-ensemble, $\bar{E}$ est compact, $\Psi : E \to \mathbb{R}^n$ telle que $\Psi \in C^1(E)$, et $\Psi : E \to \Psi(E) = D$ bijective. Soit $f : \bar{D} = \overline{\Psi(E)} \to \mathbb{R}$ fonction continue.

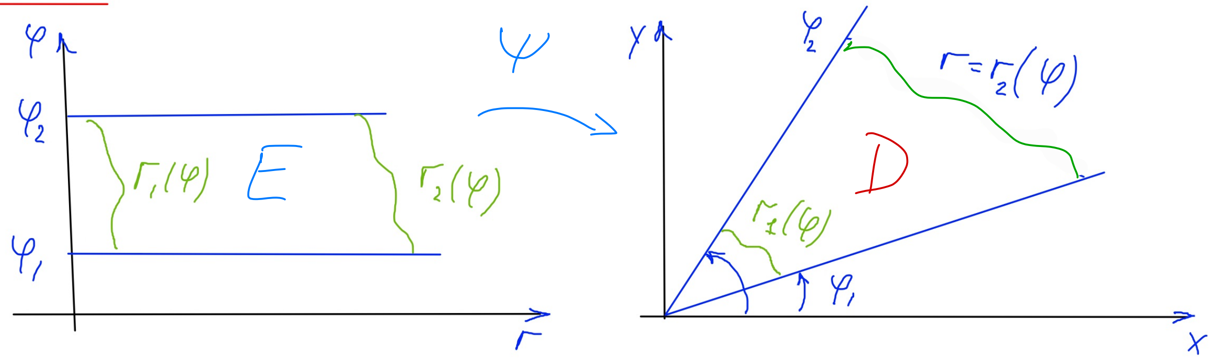
Alors
$$\boxed{\int_D f(\bar{x})\,d\bar{x} = \int_E f(\Psi(\bar{u}))\,\underline{|\det J_\Psi(\bar{u})|}\,d\bar{u}}$$



Cas important : Coordonnées polaires.

$$G(r,\varphi) = \begin{cases} r\cos\varphi = x \\ r\sin\varphi = y \end{cases}$$

$$\underset{r}{]0,\infty[} \times \underset{\varphi}{[0,2\pi[} \xrightarrow{G} \underset{(x,y)}{\mathbb{R}^2 \setminus \{\bar{0}\}}$$



$$J_G(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\Rightarrow |\det J_G(r,\varphi)| = r \quad \Rightarrow$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{E = G^{-1}(D)} f(r\cos\varphi, r\sin\varphi)\cdot \underbrace{r}_{|J_G|}\,dr\,d\varphi$$

Coordonnées spheriques.

$$G(r,\vartheta,\varphi) = \begin{cases} x = r\sin\vartheta\cos\varphi \\ y = r\sin\vartheta\sin\varphi \\ z = r\cos\vartheta \end{cases}$$

$$G : \underset{r}{]0,\infty[} \times \underset{\vartheta}{[0,\pi]} \times \underset{\varphi}{[0,2\pi[} \longrightarrow \underset{(x,y,z)}{\mathbb{R}^3 \setminus \{\bar{0}\}}$$

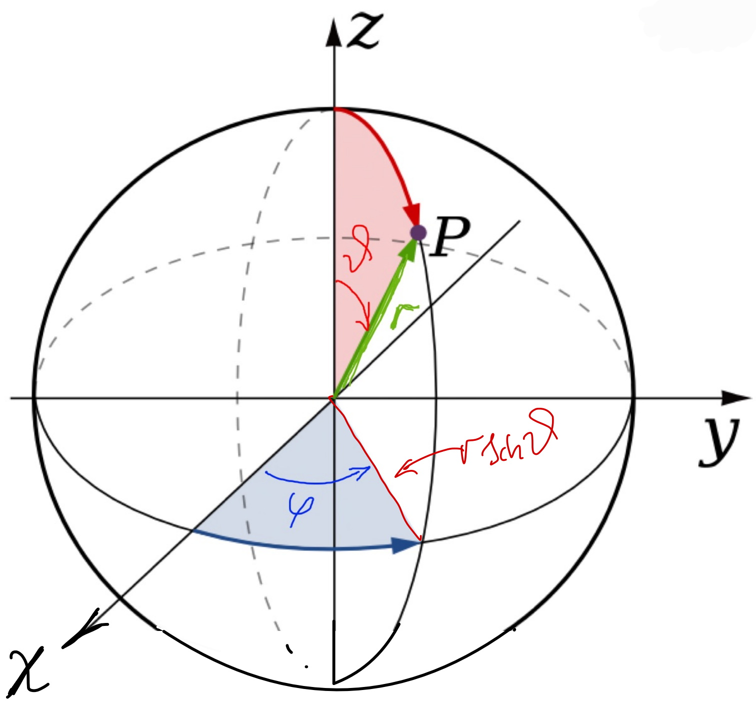
$$r = \sqrt{x^2+y^2+z^2}$$

$$\vartheta = \arccos\frac{z}{r},\ r>0$$

Si $\sin\vartheta \neq 0 \Rightarrow$

$$\cos\varphi = \frac{x}{r\sin\vartheta}$$

$$\sin\varphi = \frac{y}{r\sin\vartheta}$$

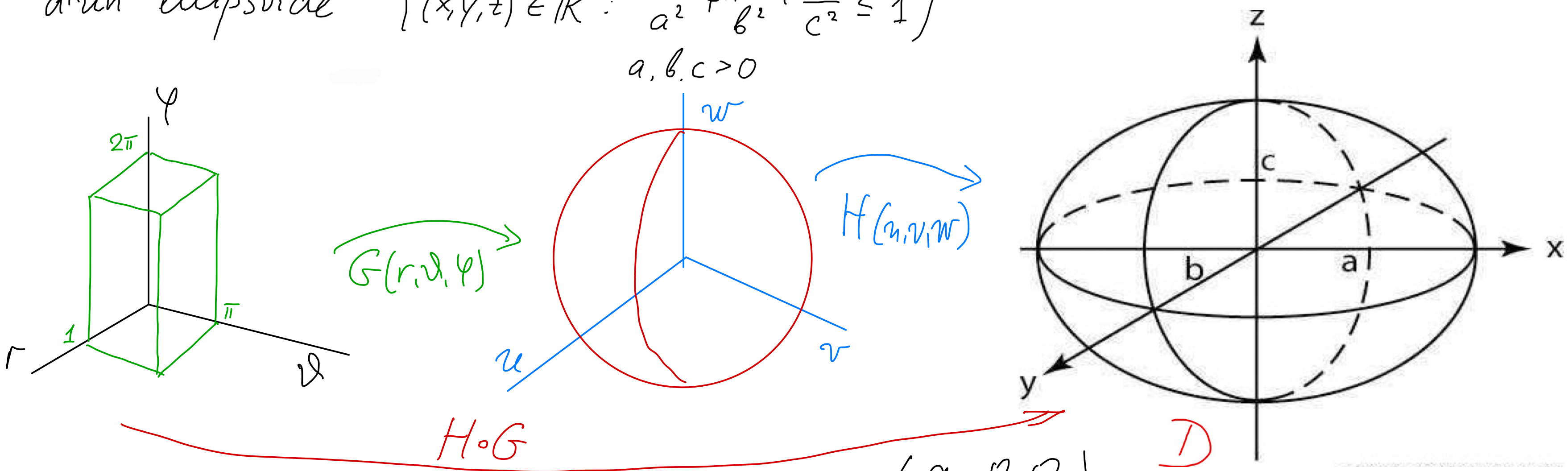


$$J_{G(r,\vartheta,\varphi)} = \begin{pmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix}$$

$$\boxed{\left|\det J_{G(r,\vartheta,\varphi)}\right| = r^2\sin\vartheta}$$

Ex Volume d'un ellipsoïde $\left\{(x,y,z)\in\mathbb{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$

$a, b, c > 0$



$$\begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \Rightarrow \left\{(u,v,w)\in\mathbb{R}^3 : u^2+v^2+w^2\le 1\right\} \Rightarrow J_{H(u,v,w)} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \Rightarrow |\det J_H| = abc$$

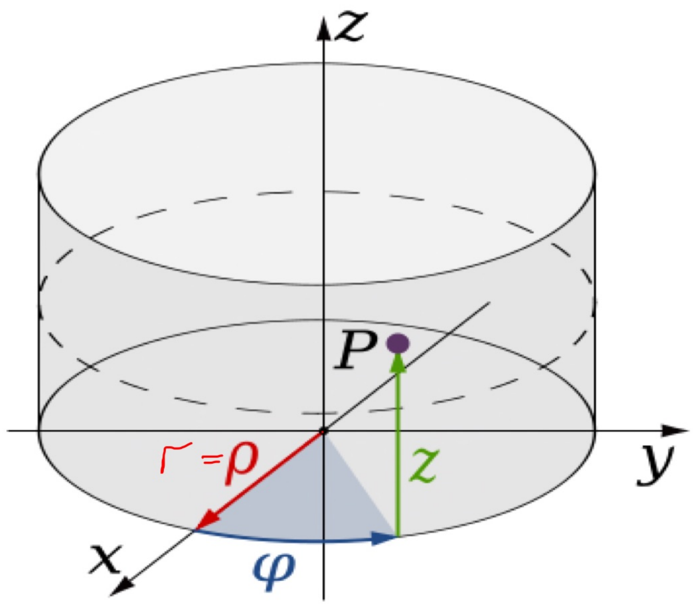
$$J_{H\circ G} = J_{H(G)}\cdot J_G \Rightarrow |\det J_{H\circ G}| = \underbrace{|\det J_{H(G)}|}_{abc}\cdot\underbrace{|\det J_G|}_{r^2\sin\vartheta} = abc\, r^2\sin\vartheta$$

$$\Rightarrow V = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \int_0^1 abc\, r^2 \sin\vartheta\, dr = abc \underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta \int_0^1 r^2\, dr}_{V(\text{sphère solide de rayon } 1)} = abc\,\frac{4}{3}\pi.$$

Coordonnées cylindriques

Coordonnées cylindriques.

$$G(r,\varphi,z) = \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \qquad G : [0,\infty[\times [0,2\pi[\times \mathbb{R} \to \mathbb{R}^3$$



$$J_{G(r,\varphi,z)} = \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \det J_{G(r,\varphi,z)} = 1 \cdot \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

Ex : Trouver le volume du domaine D avec les frontières

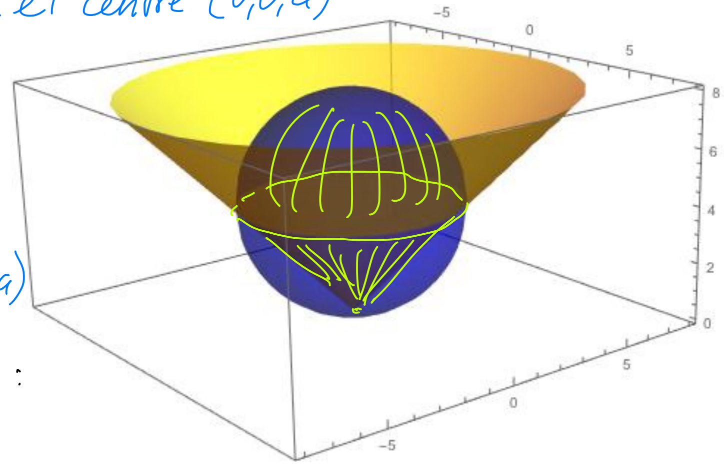
$$\begin{cases} x^2+y^2+z^2 = 2za, \quad a>0 \\ x^2+y^2 = z^2 \end{cases}, \quad \text{qui contient } (0,0,a).$$

=> en coordonnées cylindriques :

sphère de rayon a et centre (0,0,a)

$$\begin{cases} r^2+z^2-2za+a^2=a^2 \\ r^2 = z^2 \end{cases} \Rightarrow \begin{cases} r^2+(z-a)^2 = a^2 \\ r = z, \quad z \geq 0 \\ r=-z, \quad z\leq 0 \end{cases}$$

le cône $r=z>0$ qui contient (0,0,a)



L'intersection : $r=z$ => dans l'équation de la sphère :

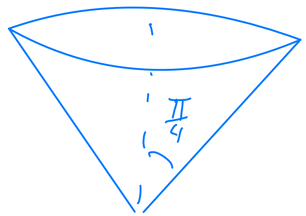
$$r^2+(r-a)^2=a^2 \Rightarrow r^2+r^2-2ra+a^2=a^2$$

$$\Rightarrow 2r^2-2ra=0 \Rightarrow 2r(r-a)=0 \Rightarrow r=a=z$$

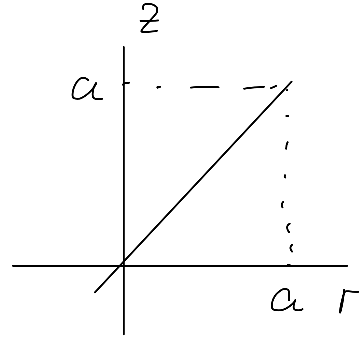
=> cercle de rayon a dans le plan $z=a$

$V = \left[\frac{1}{2}\text{ sphère solide de rayon } a\right] + \left[\text{cône } r=z \text{ coupé par } 0<z<a\right]$

$$\frac{1}{2}\cdot\frac{4}{3}\pi a^3 = \frac{2}{3}\pi a^3$$



Le volume du cône : $\{0 < z < a,\ 0 < \varphi < 2\pi,\ 0 < r < z\}$



$$V_{\text{Cône}} = \int_0^{2\pi} d\varphi \int_0^a dz \int_0^z r\, dr = 2\pi \int_0^a \left(\frac{1}{2} r^2 \Big|_0^z \right) dz = \pi \int_0^a z^2 dz =$$

$|\det J_G|$

$$= \frac{1}{3} \pi z^3 \Big|_0^a = \frac{1}{3} \pi a^3$$

$$V_D = V_{\frac{1}{2}\text{sphère}} + V_{\text{Cône}} = \frac{2}{3}\pi a^3 + \frac{1}{3}\pi a^3 = \underline{\pi a^3}$$

<u>Exercice</u>: Calculer le volume de $\frac{1}{2}$ sphere $U = \begin{cases} x^2+y^2+(z-a)^2 \leq a^2 \\ z \geq a \end{cases}$

en coordonnées cylindriques $G^{-1}(U) = \{(r, \varphi, z) : 0 < \varphi < 2\pi,\ 0 < r < a,\ a < z < a + \sqrt{a^2 - r^2}\}$

$r^2 + (z-a)^2 = a^2 \Rightarrow z = a + \sqrt{a^2 - r^2}$

$$V_{1/2\,\text{sphère}} = \int_0^{2\pi} d\varphi \int_0^a dr \int_a^{a+\sqrt{a^2-r^2}} r\, dz = 2\pi \int_0^a dr \left(r \left(a + \sqrt{a^2 - r^2} - a \right) \right) = 2\pi \cdot \frac{1}{2} \int_0^a \sqrt{a^2 - r^2}\, dr^2 = -\frac{2\pi}{3} (a^2 - r^2)^{3/2} \Big|_0^a =$$

$$= \underline{\frac{2}{3} \pi a^3}.$$

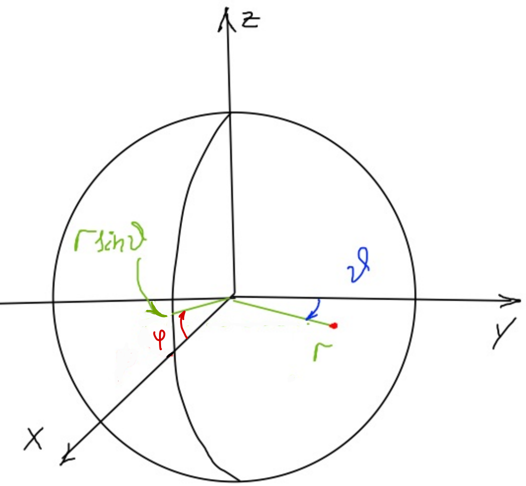
Géometrie du changement de variables

Remarque. Dans le changement de variables sphériques et cylindriques, une coordonnée cartesienne est de la forme speciale: z. Mais selon la géométrie du domaine et la fonction donnée, on peut choisir une autre coordonnée cartesienne d'avoir cette forme speciale, par exemple y.

$$G_{sph} = \begin{cases} x = r\sin\vartheta\cos\varphi \\ z = r\sin\vartheta\sin\varphi \\ y = r\cos\vartheta \end{cases}$$

$$\underset{r}{[0,\infty[} \times \underset{\vartheta}{[0,\pi]} \times \underset{\varphi}{[0,2\pi[}$$

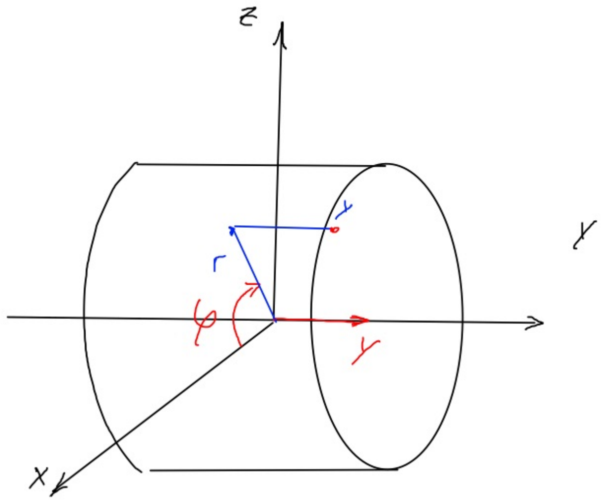
$$\boxed{|J_{G_{sph}}| = r^2\sin\vartheta}$$



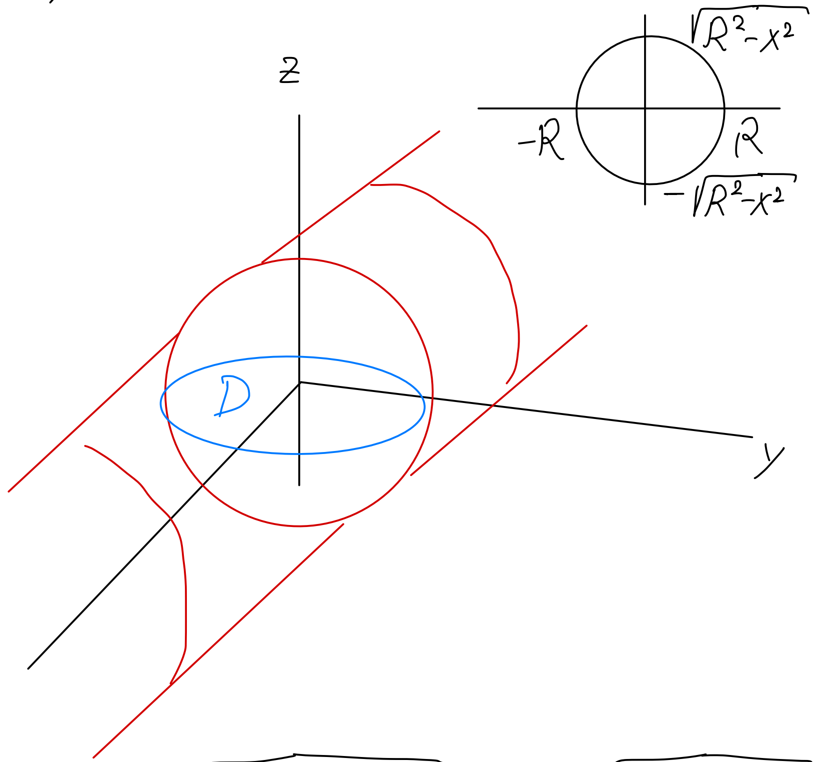
$$G_{cyl} = \begin{cases} x = r\cos\varphi \\ z = r\sin\varphi \\ y = y \end{cases}$$

$$\underset{r}{[0,\infty[} \times \underset{\varphi}{[0,2\pi[} \times \underset{y}{\mathbb{R}}$$

$$\boxed{|J_{G_{cyl}}| = r}$$



Question 21(B): Quel est l'objet geometrique dont le volume est donné par l'intégrale



A. $$\underbrace{\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz$$

intersection de deux cylindres

$$D=\left\{-R<x<R,\ -\sqrt{R^2-x^2}<y<\sqrt{R-x^2}\right\}$$

$$E=\left\{(x,y)\in D,\ -\sqrt{R^2-y^2}<z<\sqrt{R^2-y^2}\right\}$$

En coordonnée cylindriques: $E=\left\{0<\varphi<2\pi,\ 0<r<R,\ -\sqrt{R^2-r^2\sin^2\varphi}<z<\sqrt{R^2-r^2\sin^2\varphi}\right\}$

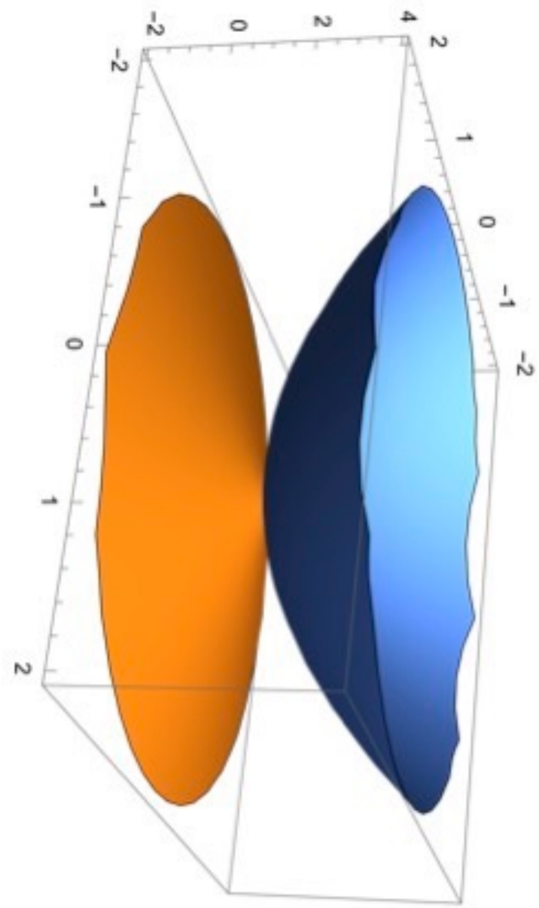
$$2\int_0^{2\pi}d\varphi\int_0^R dr\int_0^{\sqrt{R^2-r^2\sin^2\varphi}} r\,dz = 2\int_0^{2\pi}d\varphi\int_0^R r\sqrt{R^2-r^2\sin^2\varphi}\,dr = 2\int_0^{2\pi}d\varphi\int_0^R r|\sin\varphi|\sqrt{\frac{R^2}{\sin^2\varphi}-r^2}\,dr =$$

$$= -\int_0^{2\pi}d\varphi\int_0^R|\sin\varphi|\left(\frac{R^2}{\sin^2\varphi}-r^2\right)^{1/2}d\left(\frac{R^2}{\sin^2\varphi}-r^2\right) = -\frac{2}{3}\int_0^{2\pi}|\sin\varphi|\left(R^3\left(\frac{1}{\sin^2\varphi}-1\right)^{3/2}-\frac{R^3}{|\sin\varphi|^3}\right)d\varphi = -\frac{2R^3}{3}\int_0^{2\pi}|\sin\varphi|\left(\frac{|\cos\varphi|^3-1}{|\sin\varphi|^3}\right)d\varphi$$

$$= -\frac{2R^3}{3}\int_0^{2\pi}\frac{|\cos\varphi|^3-1}{\sin^2\varphi}d\varphi = -\frac{4R^3}{3}\int_0^{2\pi}\frac{\cos^3\varphi-1}{\sin^2\varphi}d\varphi = \dots = \frac{16R^3}{3}.$$

Ex. $\iiint_E e^y\,dx\,dy\,dz$, $E=\{(x,y,z)\in\mathbb{R}^3 : x^2+z^2\le 4,\ \underbrace{-\sqrt{x^2+z^2}}_{-r}<y<\underbrace{(x^2+z^2)}_{r^2}\}$

$\begin{cases} x = r\cos\varphi \\ z = r\sin\varphi \\ y = y \end{cases}$ Cylindrique. $|J_G| = r$



$E=\{(r,\varphi,y) : 0<\varphi<2\pi,\ 0<r<2,\ -r<y<r^2\}$

$$\int_0^{2\pi}d\varphi\int_0^2 dr\int_{-r}^{r^2}\underset{|\det J|}{r}\,e^y\,dy = 2\pi\int_0^2 dr\left(r e^y\Big|_{-r}^{r^2}\right) =$$

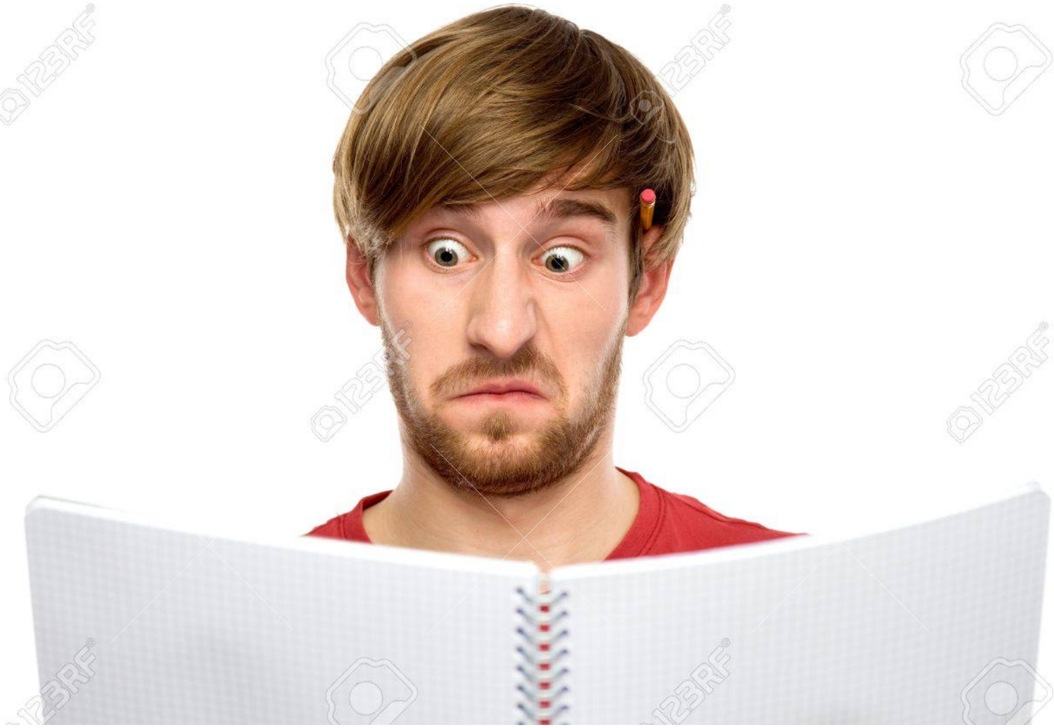
$$= 2\pi\int_0^2\left(re^{r^2}-re^{-r}\right)dr = 2\pi\int_0^2\frac{1}{2}e^{r^2}dr^2 + 2\pi\int_0^2\underbrace{r}_{f}\,d\underbrace{e^{-r}}_{g} =$$

$$= \pi e^{r^2}\Big|_0^2 + 2\pi r e^{-r}\Big|_0^2 - 2\pi\int_0^2 e^{-r}dr = \pi(e^4-1) + 4\pi e^{-2} +$$

$$+ 2\pi e^{-r}\Big|_0^2 = \pi(e^4-1)+4\pi e^{-2}+2\pi e^{-2}-2\pi = \underline{\pi e^4+6\pi e^{-2}-3\pi}$$

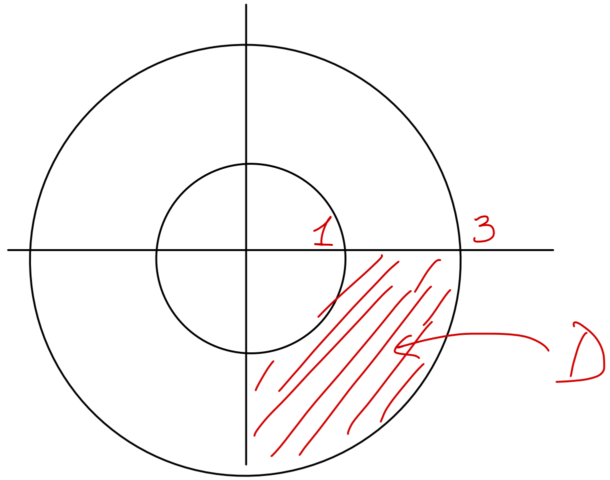
* * * Fin du cours :-)

Révisions



(1) $$\iint_D \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right) dx\,dy$$ où $D=\{(x,y)\in\mathbb{R}^2 : x\geq 0,\ y\leq 0,\ 1\leq x^2+y^2\leq 9\}$

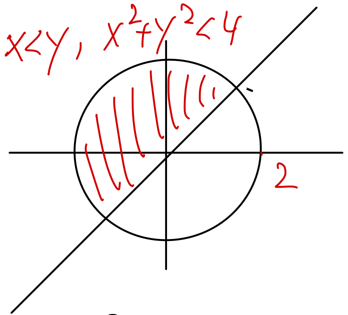
$\Rightarrow$ polaires $D=\{(r,\varphi) : -\frac{\pi}{2}<\varphi<0,\ 1<r<3\}$

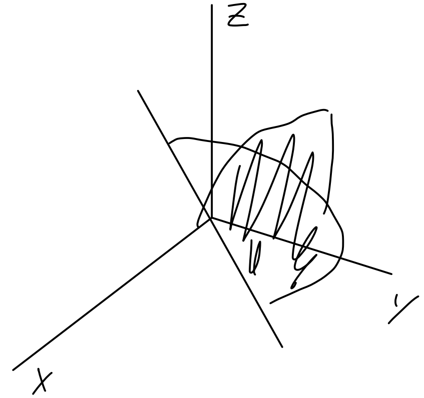


$$\rightarrow \int_{-\frac{\pi}{2}}^{0} d\varphi \int_{1}^{3} \arcsin\left(\frac{r\sin\varphi}{r}\right)\cdot \underset{|\det J|}{\underset{\uparrow}{r}}\,dr = \int_{-\frac{\pi}{2}}^{0} d\varphi \int_{1}^{3} \varphi\, r\, dr =$$

$$= \frac{1}{2}\varphi^2\Big|_{-\frac{\pi}{2}}^{0} \cdot \frac{1}{2}r^2\Big|_{1}^{3} = -\frac{1}{2}\left(\frac{\pi}{2}\right)^2\cdot\frac{1}{2}(9-1) = -\frac{1}{2}\pi^2$$

(2) $\displaystyle\iiint\limits_V z^2\,dx\,dy\,dz \qquad V=\left\{(x,y,z):\ x<y,\ x^2+y^2<4,\ 0<z<\sqrt[4]{x^2+y^2}\right\}$





Cylindriques:

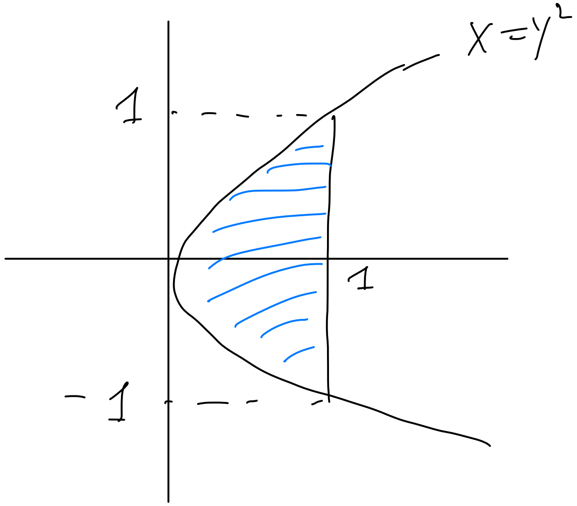
$$V=\left\{(r,\varphi,z):\ \frac{\pi}{4}<\varphi<\frac{5\pi}{4},\ 0<r<2,\ 0<z<\sqrt{r}\right\}$$

$$\iiint\limits_V z^2\,dx\,dy\,dz=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}d\varphi\int_0^2 dr\int_0^{\sqrt{r}} z^2\underbrace{r}_{|\det J|}\,dz=\varphi\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\cdot\int_0^2 r\cdot\frac{1}{3}z^3\Big|_0^{\sqrt{r}}dr=$$

$$=\pi\cdot\frac{1}{3}\int_0^2 r^{\frac{5}{2}}\,dr=\frac{\pi}{3}\cdot\frac{2}{7}r^{\frac{7}{2}}\Big|_0^2=\frac{2\pi}{21}\cdot 8\sqrt{2}=\frac{16\pi}{21}\sqrt{2}$$

Intégrales

(3) $$\iint_D \sin y \cos x \, dx\, dy \qquad D = \{(x,y)\in\mathbb{R}^2 : -1\le y\le 1,\ y^2\le x\le 1\}$$



$$= \int_{-1}^{1} dy \int_{y^2}^{1} \sin y \cos x \, dx = \int_{-1}^{1} dy \left(\sin y \left(\sin 1 - \sin(y^2)\right)\right) \quad :($$

$$D = \{(x,y)\in\mathbb{R}^2 : 0<x<1,\ -\sqrt{x}<y<\sqrt{x}\}$$

$$\int_0^1 dx \int_{-\sqrt{x}}^{\sqrt{x}} \sin y \cos x \, dy = \int_0^1 dx \left(\cos x \left(-\cos y\right)\Big|_{-\sqrt{x}}^{\sqrt{x}}\right) = \int_0^1 \cos x \underbrace{\left(-\cos\sqrt{x} + \underbrace{\cos(-\sqrt{x})}_{\cos\sqrt{x}}\right)}_{=0} dx = 0$$

:)

$f(x,y) = \sin y \cos x$

$f(x,-y) = -\sin y \cos x$, D symétrique par $y \to -y$

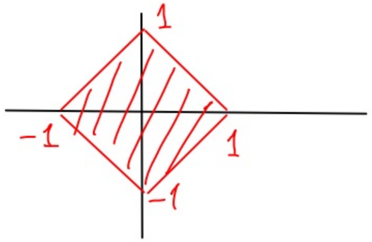
$$\Rightarrow \iint = 0.$$

Question 23

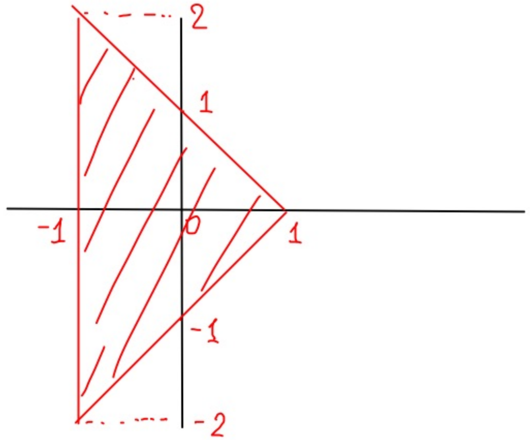
Soit $D = \{(x,y) \in \mathbb{R}^2 : \ x \geq -1, \quad |y| \leq 1 - x\}$.

Alors D est de la forme :

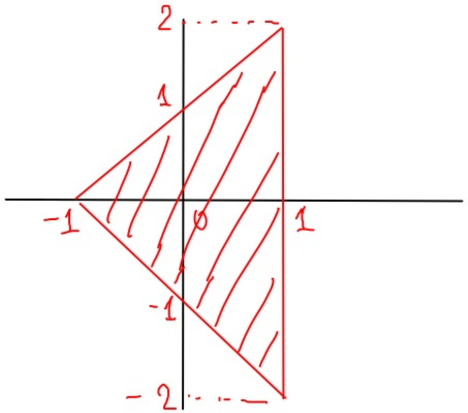
A



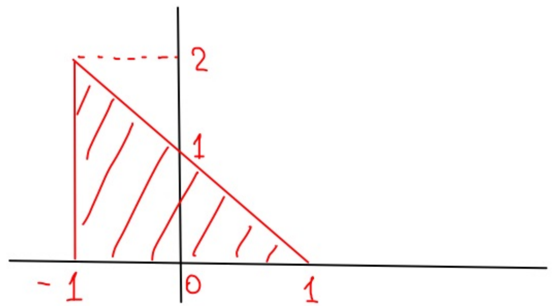
C



B



D

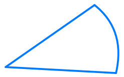


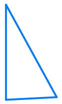
$$x \geq -1 \ \text{ et } \ \begin{cases} y \geq 0 \Rightarrow y \leq 1 - x \\ y < 0 \Rightarrow -y \leq 1 - x \Rightarrow y \geq x - 1 \end{cases} \Rightarrow C$$

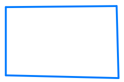
Question 24

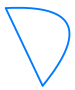
en coordonnées polaires

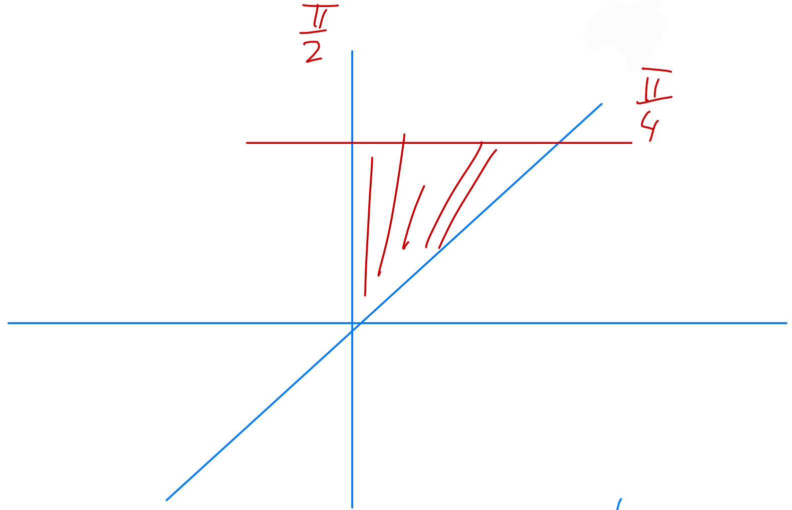
L'intégrale $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_{0}^{\frac{1}{\sin\varphi}} r\,dr$ exprime l'aire d'un(e)

A. secteur circulaire 

B. triangle 

C. rectangle 

D. tranche d'un cercle. 



$0 < r < \frac{1}{\sin\varphi}$ $\quad \sin\varphi > 0$

$\frac{\pi}{4} < \varphi < \frac{\pi}{2}$

$r\sin\varphi < 1$

$y < 1$ $\rightarrow$ triangle !