Examen final: 16 juin, 9:15 – 12:45

Salles CE 1515, CE 14, CE 15, CE 16, PO 01

- Sans documents, sans outils éléctroniques
- 80 % partie commune
- 20 % partie spécifique à la section

Rappel:

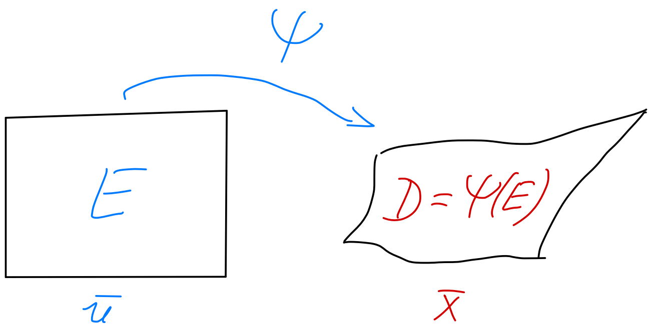
§5.5. Changement de variables dans une intégrale multiple

Thm. Soit $E \subset \mathbb{R}^n$ un sous-ensemble ($\bar{E}$ compact); $\varphi : E \to \mathbb{R}^n$ telle que $\varphi \in C^1(E)$ et $\varphi : E \to \varphi(E)$ est bijective ($J_\varphi(\bar{u})$ est inversible $\forall \bar{u} \in E$).

Soit $f : \bar{D} = \overline{\varphi(E)} \to \mathbb{R}$ fonction continue.

Alors

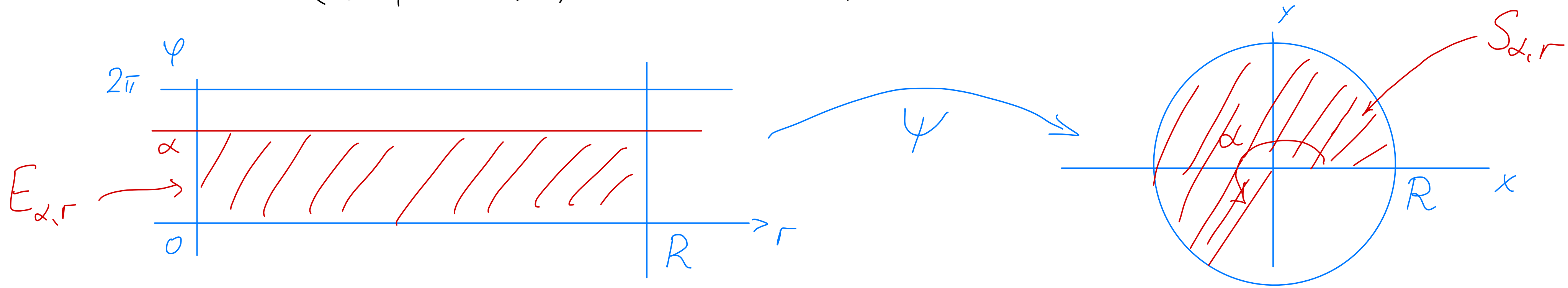
$$\boxed{\int_D f(\bar{x})\, d\bar{x} = \int_E f(\varphi(\bar{u}))\, |\det J_\varphi(\bar{u})|\, d\bar{u}}$$



Application: changement de variables polaires.

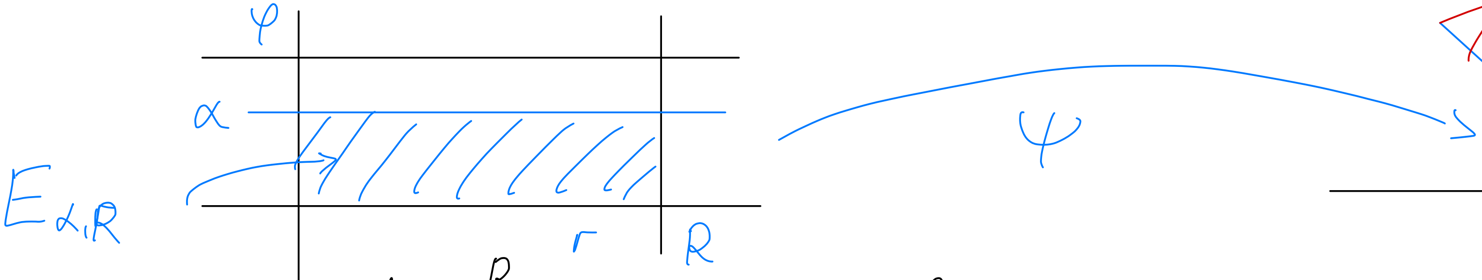
$$\psi(r,\varphi) = \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \psi : \underset{r}{\mathbb{R}_+} \times \underset{\varphi}{[0,2\pi[} \xrightarrow[\text{bijective}]{} \mathbb{R}^2 \setminus \{0\}.$$

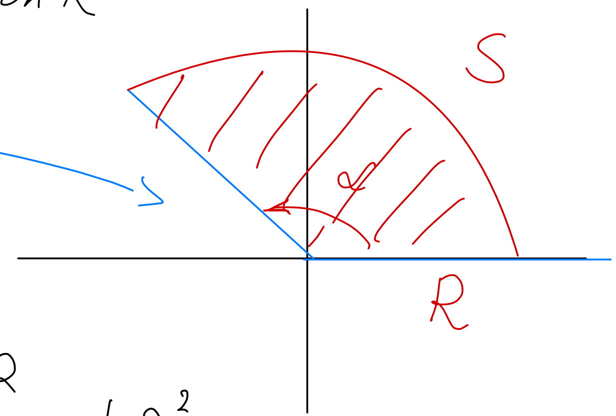
$$J_\psi(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \Rightarrow \det J_{\psi(r,\varphi)} = r(\cos^2\varphi + \sin^2\varphi) = r$$



Ex. 1

Trouver l'aire du secteur circulaire $S_{\alpha,r}$ d'angle α et rayon R





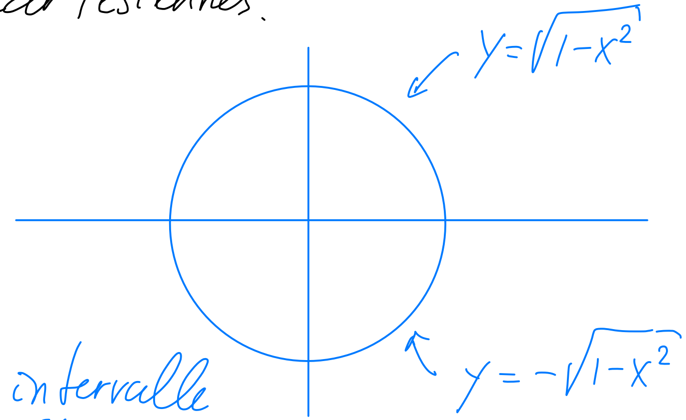
$$A_{\alpha,R} = \iint_S 1\,dx\,dy = \int_0^\alpha d\varphi \int_0^R |\det \psi_{r,\varphi}|\,dr = \int_0^\alpha d\varphi \int_0^R r\,dr = \alpha \cdot \frac{1}{2} r^2 \Big|_0^R = \frac{1}{2} R^2 \alpha$$

Si $\alpha = 2\pi \Rightarrow$ L'aire du disque de rayon R est

$$A_{2\pi, R} = \frac{1}{2} R^2 \cdot 2\pi = \pi R^2$$

Exercice: L'aire du disque de rayon 1 en coordonnées cartesiennes.

$$\mathcal{D} = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}$$



$$\text{Aire} = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy = \int_{-1}^{1} 2\sqrt{1-x^2}\, dx = 4\int_{0}^{1} \sqrt{1-x^2}\, dx =$$

$x = \sin t$

fonction paire sur un intervalle symétrique

$$= 4\int_0^1 \sqrt{1-x^2}\, dx = \left[\text{Changement de variable} \right] = 4 \int_0^{\frac{\pi}{2}} \underbrace{\sqrt{1-\sin^2 t}}_{|\cos t| = \cos t \text{ sur } [0, \frac{\pi}{2}]} \, d\sin t = 4 \int_0^{\frac{\pi}{2}} \cos t \cdot \cos t \, dt =$$

$x = \sin t$

$[0,1] \longrightarrow [0, \frac{\pi}{2}]$

$$= 4 \int_0^{\frac{\pi}{2}} \cos^2 t \, dt = 2 \int_0^{\frac{\pi}{2}} (1 + \cos 2t)\, dt = 2 \cdot t \Big|_0^{\frac{\pi}{2}} + \sin 2t \Big|_0^{\frac{\pi}{2}} = 2 \cdot \frac{\pi}{2} + 0 = \underline{\pi}$$

$$2\cos^2 t - 1 = \cos 2t$$

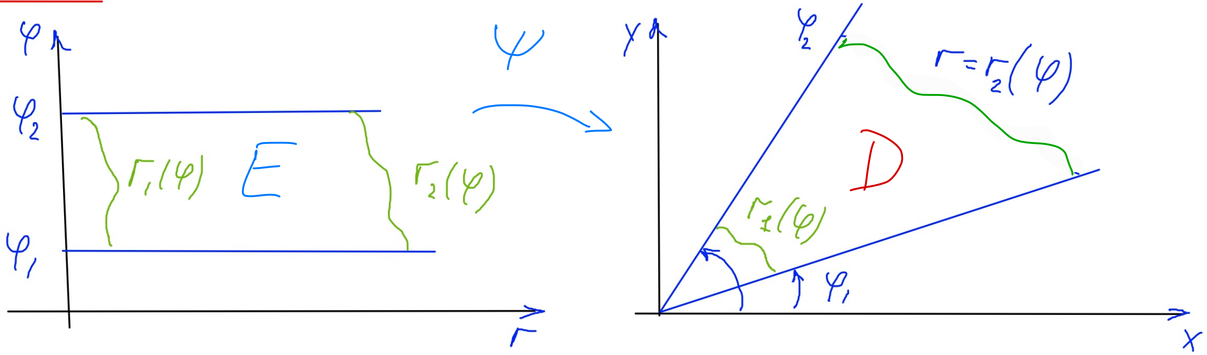
$$A_{2\pi, 1} = \pi \cdot 1^2 = \pi.$$

Conclusion: changement de variables polaires.

Cas important : Coordonnées polaires.

$$G(r,\varphi) = \begin{cases} r\cos\varphi = x \\ r\sin\varphi = y \end{cases}$$

$$\underset{r}{]0,\infty[} \times \underset{\varphi}{[0,2\pi[} \xrightarrow{G} \underset{(x,y)}{\mathbb{R}^2 \setminus \{\bar{0}\}}$$



$$J_G(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\Rightarrow \quad |\det J_G(r,\varphi)| = r \quad \Rightarrow$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{E = G^{-1}(D)} f(r\cos\varphi, r\sin\varphi) \cdot \underbrace{r}_{|J_G|}\, dr\, d\varphi$$

Ex 2 $f(x,y)=\sqrt{a^2-x^2-y^2}$; D est une boucle de la lemniscate de Bernoulli (1694)
$a>0$

$$\underbrace{(x^2+y^2)}_{r^2}{}^2 = a^2(x^2-y^2)\ ,\quad \underline{x>0}.$$

En coordonnées polaires : $r^4 = a^2(r^2\cos^2\varphi - r^2\sin^2\varphi) = a^2r^2\cos 2\varphi$

$$\Leftrightarrow\quad r^2 = a^2\cos 2\varphi \quad \Rightarrow \cos 2\varphi \geq 0$$

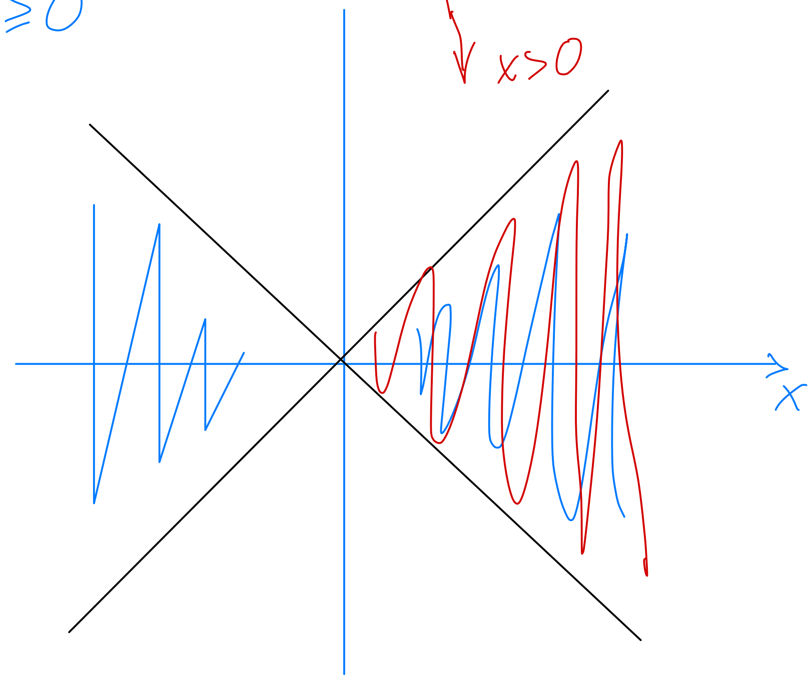
$$\cos 2\varphi > 0 \Rightarrow -\frac{\pi}{2} < 2\varphi < \frac{\pi}{2}$$

$$\Rightarrow \begin{cases} -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4} & \underline{x>0} \\ \frac{3\pi}{4} < \varphi < \frac{5\pi}{4} & x<0 \end{cases}$$

$$\Rightarrow \underline{r = a\sqrt{\cos 2\varphi}}\ ,\quad -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$$

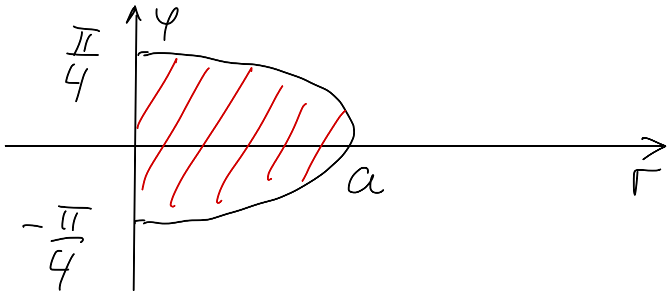
L'équation d'une boucle de la lemniscate

$\varphi = \pm\frac{\pi}{4} \Rightarrow r=0$, $\varphi = 0 \Rightarrow r = a$

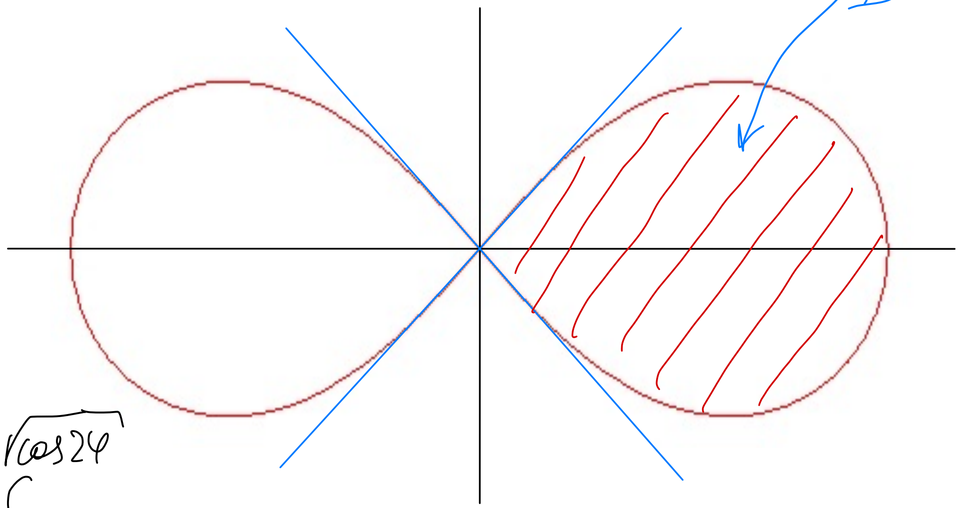


$$E = \left\{-\frac{\pi}{4} < \varphi < \frac{\pi}{4}\ ,\ 0 < r < a\sqrt{\cos 2\varphi}\right\} \xrightarrow{\Psi(r,\varphi)} D$$

$$E=\left\{-\frac{\pi}{4}<\varphi<\frac{\pi}{4},\ 0<r<a\sqrt{\cos 2\varphi}\right\}$$



$$\xrightarrow{\Psi(r,\varphi)}$$



$$\iint\limits_D \sqrt{a^2-x^2-y^2}\,dx\,dy=\iint\limits_E \sqrt{a^2-r^2}\,\underset{|\det J|}{r}\,dr\,d\varphi=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\varphi\int_0^{a\sqrt{\cos 2\varphi}}\sqrt{a^2-r^2}\,r\,dr$$

$$-\frac{1}{2}\int_0^{a\sqrt{\cos 2\varphi}}\sqrt{a^2-r^2}\,d(a^2-r^2)=-\frac{1}{2}\cdot\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}\Big|_0^{a\sqrt{\cos 2\varphi}}=-\frac{1}{3}\left(a^2-a^2\cos 2\varphi\right)^{3/2}+\frac{1}{3}a^3=$$

$$d(a^2-r^2)=-2r\,dr$$

$$=-\frac{1}{3}a^3 2^{3/2}|\sin\varphi|^3+\frac{1}{3}a^3$$

$$(1-\cos 2\varphi)^{3/2}=\left(1-\cos^2\varphi+\sin^2\varphi\right)^{3/2}=2^{3/2}(\sin^2\varphi)^{3/2}=2^{3/2}|\sin\varphi|^3$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\varphi$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(-\frac{1}{3}2^{3/2}a^3|\sin\varphi|^3+\frac{1}{3}a^3\right)d\varphi=\frac{1}{3}a^3\left(\frac{\pi}{4}+\frac{\pi}{4}\right)-\frac{1}{3}a^3 2^{3/2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}|\sin\varphi|^3 d\varphi=$$

fonction paire sur un intervalle symétrique

$$=\frac{\pi}{6}a^3-\frac{1}{3}a^3 2^{3/2}\cdot 2\int_0^{\frac{\pi}{4}}\sin^3\varphi\, d\varphi$$

$$\int_0^{\frac{\pi}{4}}\sin^3\varphi\, d\varphi=\int_0^{\frac{\pi}{4}}(1-\cos^2\varphi)\,d(-\cos\varphi)=\frac{1}{3}\cos^3\varphi-\cos\varphi\Big|_0^{\frac{\pi}{4}}=\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3-\frac{1}{\sqrt{2}}-\frac{1}{3}+1=$$

$$=\frac{2}{3}+\frac{1}{\sqrt{2}}\left(\frac{1}{6}-1\right)=\frac{2}{3}-\frac{5}{6\sqrt{2}}$$

$$\iint=\frac{\pi a^3}{6}-\frac{a^3 2^{3/2}\cdot 2}{3}\left(\frac{2}{3}-\frac{5}{6\sqrt{2}}\right)=\underline{\frac{a^3}{3}\left(\frac{\pi}{2}-\frac{8\sqrt{2}-10}{3}\right)>0.}$$

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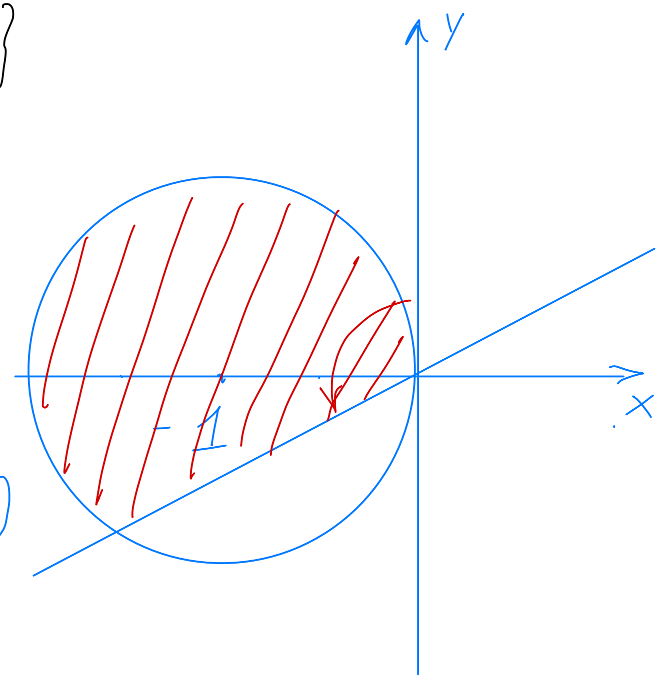
Ex. 3 $\quad f(x,y)=xy \quad , \quad D=\left\{(x+1)^2+y^2\le 1,\ y\ge \frac{x}{\sqrt{3}}\right\}$



$$r^2\cos^2\varphi+2r\cos\varphi+1+r^2\sin^2\varphi\le 1 \Rightarrow r^2+2r\cos\varphi\le 0$$

$$\underset{r>0}{\Rightarrow}\quad r+2\cos\varphi\le 0 \Rightarrow r\le -2\cos\varphi \Rightarrow \cos\varphi<0 \Rightarrow \frac{\pi}{2}<\varphi<\frac{3\pi}{2}$$

$$y\ge\frac{x}{\sqrt{3}} \Rightarrow r\sin\varphi\ge\frac{r\cos\varphi}{\sqrt{3}} \underset{r>0}{\Rightarrow} \sin\varphi\ge\frac{\cos\varphi}{\sqrt{3}} \Rightarrow \begin{cases}\sin\varphi>0,\cos\varphi<0 \Rightarrow \frac{\pi}{2}<\varphi<\pi\\ \sin\varphi<0,\cos\varphi<0 \Rightarrow \tan\varphi<\frac{1}{\sqrt{3}}\end{cases}$$

$$\tan\varphi=\frac{1}{\sqrt{3}} \Rightarrow \varphi=\frac{\pi}{6}+\pi k \qquad \Rightarrow \pi<\varphi<\frac{7}{6}\pi.$$

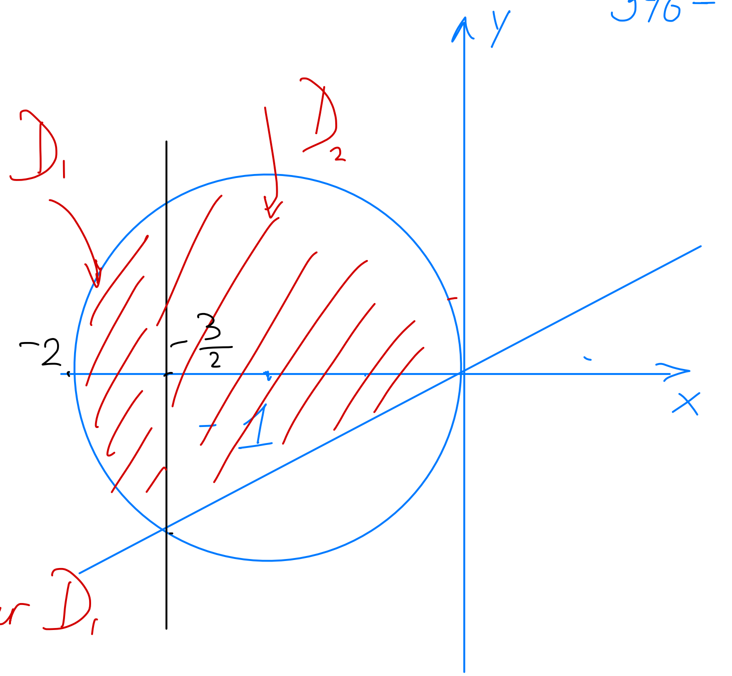
$$\Rightarrow E=\left\{\frac{\pi}{2}<\varphi<\frac{7}{6}\pi,\ 0<r<-2\cos\varphi\right\}$$

$$\iint_D xy\,dx\,dy=\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}}d\varphi\int_0^{-2\cos\varphi}r^2\cos\varphi\sin\varphi\,\underbrace{r}_{|\det J|}\,dr=\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}}d\varphi\left(\frac{1}{4}r^4\cos\varphi\sin\varphi\right)\Big|_0^{-2\cos\varphi}=4\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}}\cos^5\varphi\sin\varphi\,d\varphi=$$

$$=-4\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}}\cos^5\varphi\,d\cos\varphi=-\frac{4}{6}\cos^6\varphi\Big|_{\frac{\pi}{2}}^{\frac{7\pi}{6}}=-\frac{2}{3}\left(\frac{3}{4}\right)^3-0=-\frac{9}{32}$$

$$\cos\frac{7\pi}{6}=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$$

En coordonnées cartesiennes:



$$D = \left\{ (x+1)^2 + y^2 \le 1,\ y \ge \frac{x}{\sqrt{3}} \right\}$$

$$y = \frac{x}{\sqrt{3}} \Rightarrow (x+1)^2 + \frac{x^2}{3} = 1 \Rightarrow \frac{4}{3}x^2 + 2x + 1 = 1 = \begin{cases} x = 0 \\ \frac{4}{3}x = -2 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ x = -\frac{3}{2} \end{cases};\quad y = -\frac{\sqrt{3}}{2};$$

$$y^2 \le 1-(x+1)^2 \Rightarrow -\sqrt{1-(x+1)^2} < y < \sqrt{1-(x+1)^2}$$ sur D_1

$$\frac{x}{\sqrt{3}} < y < \sqrt{1-(x+1)^2}$$ sur D_2

$$D = \left\{ -2 \le x \le -\frac{3}{2},\ -\sqrt{1-(x+1)^2} < y < \sqrt{1-(x+1)^2} \right\} \cup \left\{ -\frac{3}{2} < x < 0,\ \frac{x}{\sqrt{3}} < y < \sqrt{1-(x+1)^2} \right\}$$

$$\iint_D xy\,dx\,dy = \int_{-2}^{-\frac{3}{2}} dx \int_{-\sqrt{1-(x+1)^2}}^{\sqrt{1-(x+1)^2}} xy\,dy + \int_{-\frac{3}{2}}^{0} dx \int_{\frac{x}{\sqrt{3}}}^{\sqrt{1-(x+1)^2}} xy\,dy = \int_{-\frac{3}{2}}^{0} dx \cdot \frac{1}{2}x\left(\left(1-(x+1)^2\right) - \frac{x^2}{3}\right) =$$

$$= \frac{1}{2}\int_{-\frac{3}{2}}^{0} \left(-\frac{4}{3}x^3 - 2x^2\right)dx = \frac{1}{2}\cdot\left(-\frac{4}{3}\right)\left(\frac{1}{4}\right)x^4 - \frac{1}{3}x^3 \Big|_{-\frac{3}{2}}^{0} = +\frac{1}{6}\,\frac{81}{16} - \frac{1}{3}\,\frac{27}{8} = \frac{27-36}{32} = -\frac{9}{32}$$

comme avant.

Coordonnées spheriques.

$$G(r,\vartheta,\varphi) = \begin{cases} x = r\sin\vartheta\cos\varphi \\ y = r\sin\vartheta\sin\varphi \\ z = r\cos\vartheta \end{cases}$$

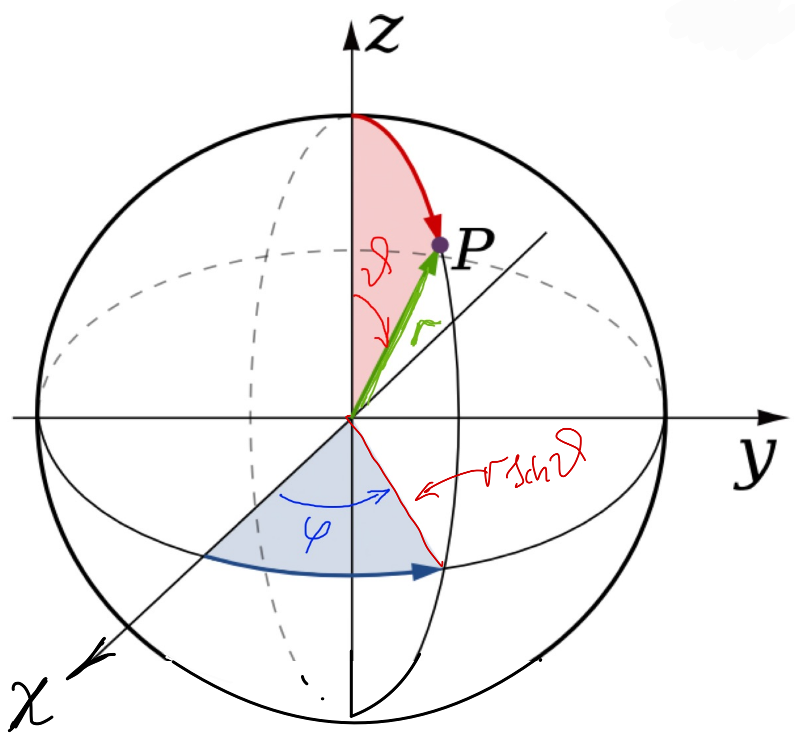
$$r = \sqrt{x^2+y^2+z^2}$$

$$\vartheta = \arccos\frac{z}{r},\ r>0$$

Si $\sin\vartheta \neq 0 \Rightarrow$

$$\cos\varphi = \frac{x}{r\sin\vartheta}$$

$$\sin\varphi = \frac{y}{r\sin\vartheta}$$



$$G: \underset{r}{]0,\infty[} \times \underset{\vartheta}{[0,\pi]} \times \underset{\varphi}{[0,2\pi[} \longrightarrow \underset{(x,y,z)}{\mathbb{R}^3 \setminus \{\vec{0}\}}$$

$$J_{G(r,\vartheta,\varphi)} = \begin{pmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix}$$

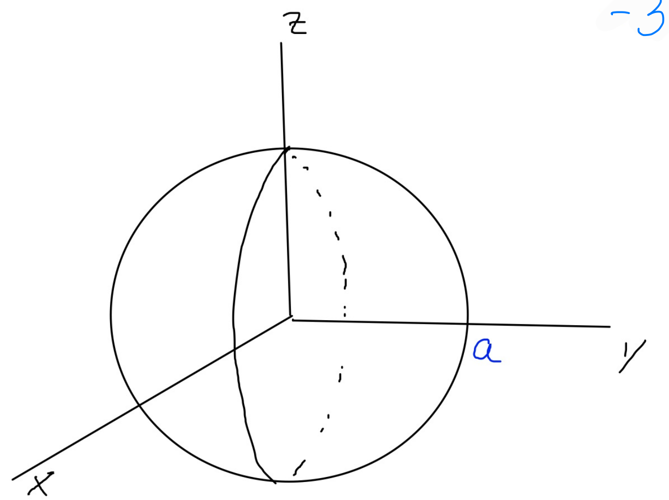
$$\Rightarrow \left|\det J_{G(r,\vartheta,\varphi)}\right| = \left| r^2\cos^2\vartheta\sin\vartheta\cos^2\varphi + r^2\sin^3\vartheta\sin^2\varphi + r^2\cos^2\vartheta\sin\vartheta\sin^2\varphi + r^2\sin^3\vartheta\cos^2\varphi \right| =$$

$$= \left| r^2\cos^2\vartheta\sin\vartheta + r^2\sin^3\vartheta \right| = \left| r^2\sin\vartheta \right| \qquad 0<\vartheta<\pi$$

$$\Rightarrow \boxed{\left|\det J_{G(r,\vartheta,\varphi)}\right| = r^2\sin\vartheta}$$

Ex. Volume d'une boule de rayon $a>0$.

$$E=\{0<r<a,\ 0<\vartheta<\pi,\ 0<\varphi<2\pi\}=B(\bar{0},a)$$



$$V=\iiint_{B(\bar{0},a)} 1\,dx\,dy\,dz=\int_0^{2\pi}d\varphi\int_0^{\pi}d\vartheta\int_0^{a}\underbrace{r^2\sin\vartheta}_{|\det J|}\,dr=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\vartheta\,d\vartheta\cdot\int_0^{a}r^2dr=2\pi\left(-\cos\vartheta\right)\Big|_0^{\pi}\cdot\frac{1}{3}r^3\Big|_0^{a}=2\pi(1+1)\cdot\frac{1}{3}a^3=\frac{4}{3}\pi a^3$$

Application : masse totale d'un objet solide

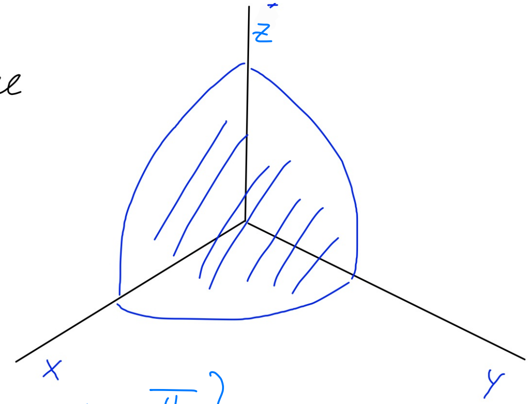
Masse totale d'un objet solide de densité donnée.

$$M=\iiint_V \underbrace{\rho(x,y,z)}_{\text{fonction densité}}\,dx\,dy\,dz$$ est la masse totale du volume V avec la densité $\rho(x,y,z)$.

Ex : Trouver la masse totale d'un secteur sphérique

$$S = \{x>0,\ y>0,\ z>0,\ x^2+y^2+z^2<a^2\}$$

avec la fonction densité $\rho(x,y,z) = x^2+y^2$



$\Rightarrow$ En coordonnées sphériques $E = \{0<r<a,\ 0<\vartheta<\frac{\pi}{2},\ 0<\varphi<\frac{\pi}{2}\}$

$$M = \iiint_S \underbrace{(x^2+y^2)}_{r^2\sin^2\vartheta}\,dx\,dy\,dz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\vartheta \int_0^a r^2\sin^2\vartheta\, \underbrace{r^2\sin\vartheta}_{|\det J|}\, dr = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin^3\vartheta\, d\vartheta \int_0^a r^4\, dr =$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (\cos^2\vartheta - 1)\, d\cos\vartheta \cdot \frac{1}{5} r^5\Big|_0^a = \frac{\pi}{2}\left(\frac{1}{3}\cos^3\vartheta - \cos\vartheta\right)\Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{5}a^5 =$$

$$= \frac{\pi a^5}{10}\left(-\frac{1}{3}+1\right) = \frac{\pi a^5}{15}$$

Question 22

$f(x,y) = 2x+3y$, $D = \{(x,y) \in \mathbb{R}^2 : x < 0,\ y > 0,\ 1 < x^2+y^2 < 4\}$

Alors $\iint_D f(x,y)\,dx\,dy$ vaut

A. $\frac{8}{3}$

B. $\frac{2}{3}$

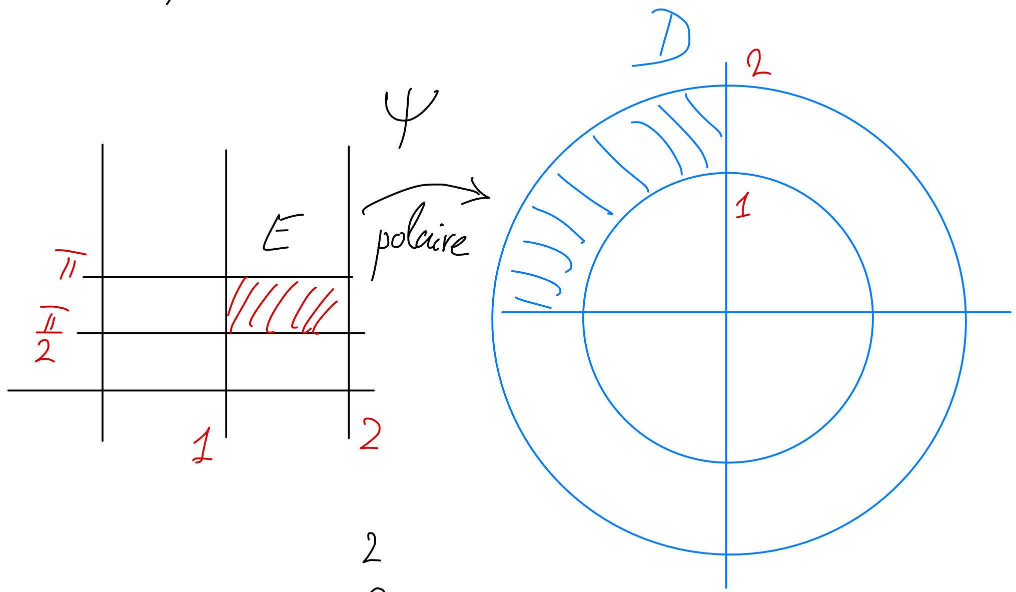
C. $\frac{4}{3}$

D. $\frac{7}{3}$

$$D = \{(x,y) \in \mathbb{R}^2 : x < 0,\ y > 0,\ 1 < x^2 + y^2 < 4\}$$

$$f(x,y) = 2x + 3y$$

$$\iint_D f(x,y)\, dx\, dy =$$



$$\iint_E (2r\cos\varphi + 3r\sin\varphi)\, \underbrace{|\det J_\psi|}_{r}\, dr\, d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\pi} d\varphi \int_1^2 r^2 (2\cos\varphi + 3\sin\varphi)\, dr = \int_{\frac{\pi}{2}}^{\pi} (2\cos\varphi + 3\sin\varphi)\, d\varphi \cdot \int_1^2 r^2\, dr =$$

$$= (2\sin\varphi - 3\cos\varphi)\Big|_{\frac{\pi}{2}}^{\pi} \cdot \frac{1}{3} r^3 \Big|_1^2 = (-2+3)\cdot\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{3}$$