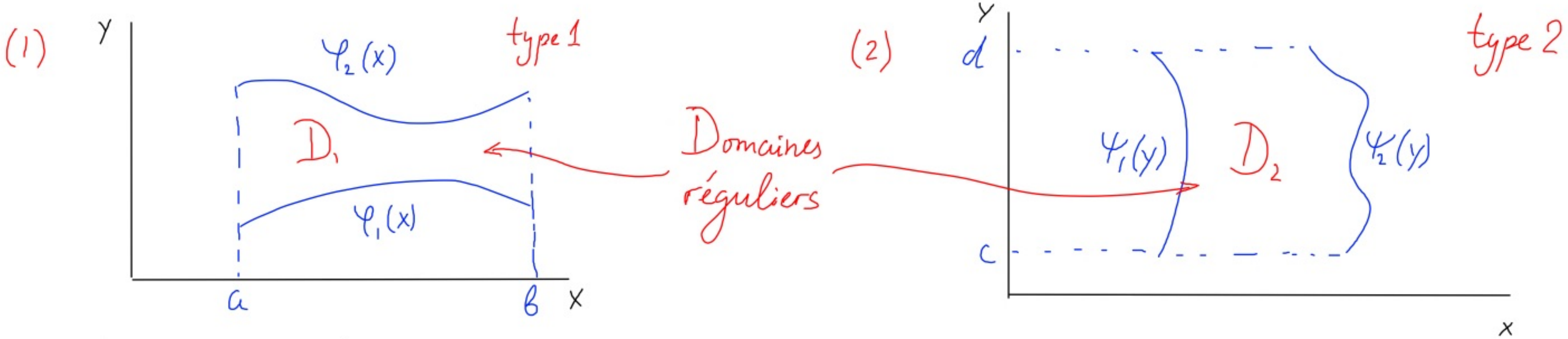
Rappel : Théorème de Fubini pour les domaines reguliers dans $\mathbb{R}^2$



$\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ continues, $\varphi_1(x) < \varphi_2(x)$ $\forall x \in]a,b[$

$f : \overline{D}_1 \to \mathbb{R}$ continue. Alors

$$\iint_{D_1} f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx$$

$\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ continues, $\psi_1(y) < \psi_2(y)$ $\forall y \in]c,d[$.

$f : \overline{D}_2 \to \mathbb{R}$ continue. Alors

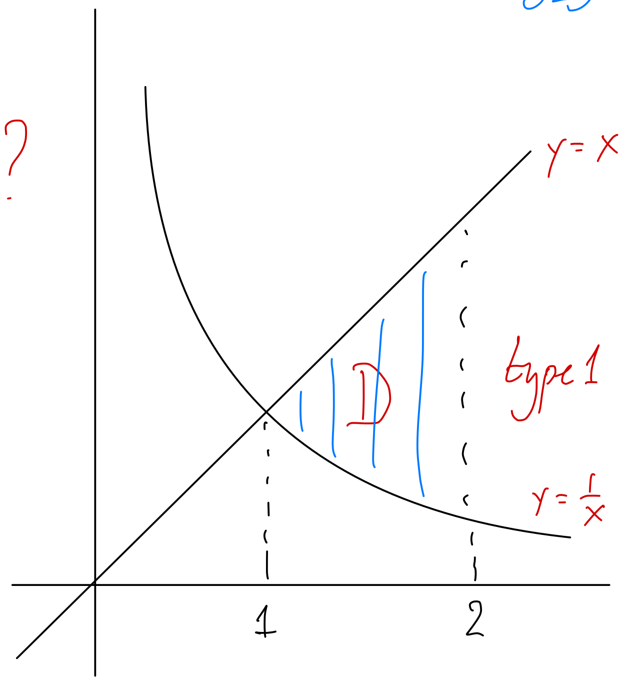
$$\iint_{D_2} f(x,y)\,dx\,dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy .$$

Si le domaine D est plus compliqué, il faut le diviser en réunion des domaines de type 1, 2, et utiliser l'additivité de l'intégrale.

Ex. 1 $D = \{(x,y) \in \mathbb{R}^2 : 1 < x < 2, \ \frac{1}{x} < y < x\}$ ← type 1

$f(x,y) = x + \frac{1}{y}$

$$\iint_D f(x,y)\,dx\,dy = ?$$



$$\int_1^2 \left(\int_{\frac{1}{x}}^{x} \left(x + \frac{1}{y}\right) dy \right) dx \overset{\text{notation}}{=} \int_1^2 dx \int_{\frac{1}{x}}^{x} \left(x + \frac{1}{y}\right) dy =$$

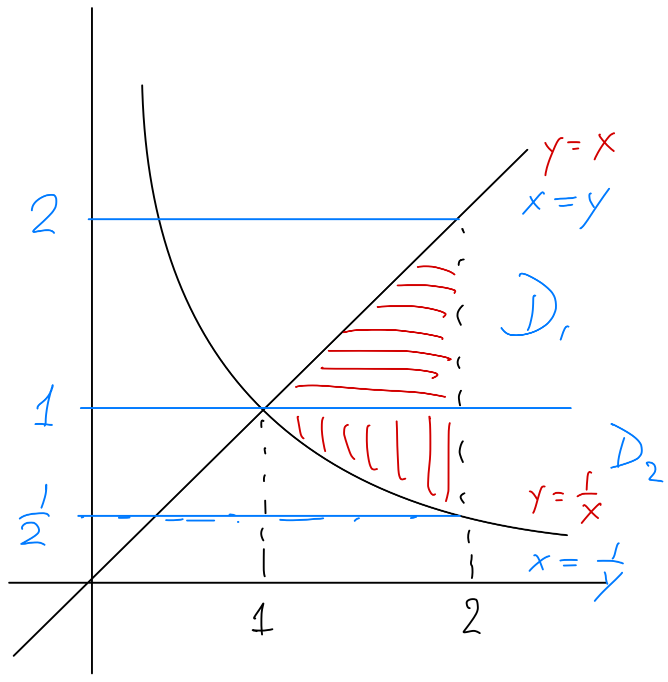
$$= \int_1^2 dx \left(xy + \ln y\right)\Big|_{\frac{1}{x}}^{x} = \int_1^2 \left(x^2 + \ln x - 1 - \overbrace{\ln\frac{1}{x}}^{-\ln x}\right) dx =$$

$$= \int_1^2 \left(x^2 + 2\ln x - 1\right) dx = \frac{1}{3}x^3 - x\Big|_1^2 + 2\int_1^2 \underbrace{\ln x}_{f}\, d\underbrace{x}_{g} = \frac{8}{3} - 2 - \frac{1}{3} + 1 + 2x\ln x\Big|_1^2 - 2\int 1\,dx =$$

$$= \frac{7}{3} - 1 + 4\ln 2 - 2x\Big|_1^2 = \frac{7}{3} - 1 + 4\ln 2 - 4 + 2 = \underline{-\frac{2}{3} + 4\ln 2} > 0.$$

Changement d'ordre d'intégration :

$$\iint_D \left(x+\frac{1}{y}\right)dx\,dy =$$



Intersection $y=x$ et $y=\frac{1}{x}$:

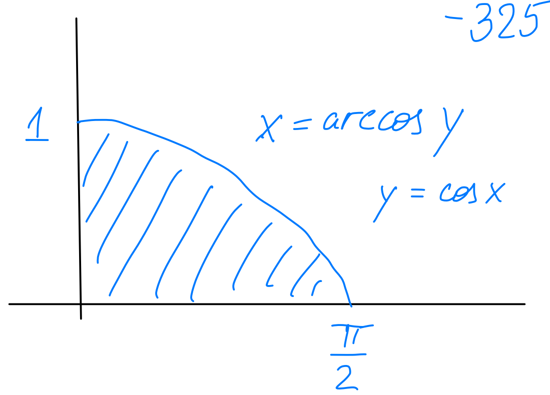
$\frac{1}{x}=x \Rightarrow x^2=1,\ x>0 \Rightarrow x=1 \Rightarrow y=1$

$D_1=\{1<y<2,\quad y<x<2\}$ type 2

$D_2=\{\frac{1}{2}<y<1,\quad \frac{1}{y}<x<2\}$ type 2

$$=\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{y}}^{2}\left(x+\frac{1}{y}\right)dx\right)dy+\int_{1}^{2}\left(\int_{y}^{2}\left(x+\frac{1}{y}\right)dx\right)dy = [\text{Exercice}] \ldots = -\frac{2}{3}+4\ln 2$$

Ex. 2 Soit $D = \{(x,y)\in\mathbb{R}^2 : 0<y<1,\ 0<x<\arccos y\}$ type 2



Calculer $\iint_D x\,dx\,dy$ $\Rightarrow D = \{(x,y)\in\mathbb{R}^2 : 0<x<\frac{\pi}{2},\ 0<y<\cos x\}$

$$\int_0^{\frac{\pi}{2}} dx \int_0^{\cos x} x\,dy = \int_0^{\frac{\pi}{2}} dx \left(xy\Big|_0^{\cos x}\right) = \int_0^{\frac{\pi}{2}} x\cos x\,dx = \int_0^{\frac{\pi}{2}} \underbrace{x}_{f}\, d\underbrace{\sin x}_{g} =$$

$$= x\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{2} - 0 + \cos x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

$y = \cos u$

$u = \arccos y$

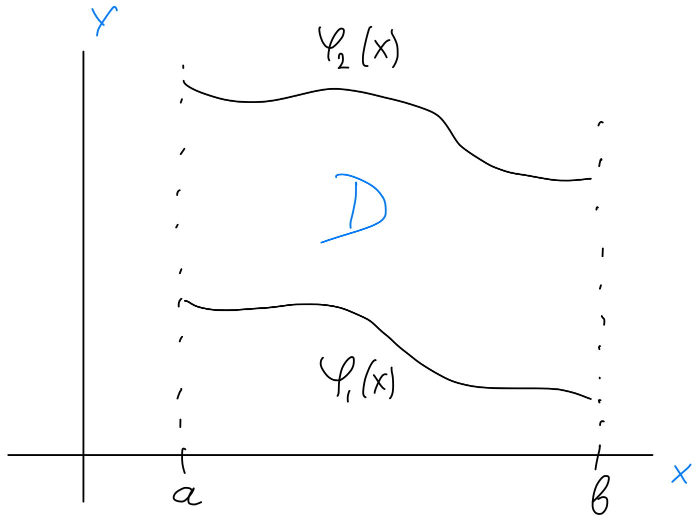
$$\iint_D x\,dx\,dy = \int_0^1 dy \int_0^{\arccos y} x\,dx = \int_0^1 dy\ \frac{1}{2}x^2\Big|_0^{\arccos y} = \frac{1}{2}\int_0^1 (\arccos y)^2\,dy = \frac{1}{2}\int_{\frac{\pi}{2}}^{0} u^2\,d\cos u =$$

$$= \frac{1}{2}u^2\cos u\Big|_{\frac{\pi}{2}}^{0} - \frac{1}{2}\int_{\frac{\pi}{2}}^{0} 2u\cos u\,du = 0 - \int_{\frac{\pi}{2}}^{0} u\,d\sin u = -u\sin u\Big|_{\frac{\pi}{2}}^{0} - \int_{\frac{\pi}{2}}^{0} \sin u\,du = \frac{\pi}{2} - \cos u\Big|_{\frac{\pi}{2}}^{0} =$$

$$= \frac{\pi}{2} - 1$$

Il vaut la peine de bien choisir la description du domaine d'intégration.

Théorème de Fubini pour les intégrales triples



Soit $[a,b]$ intervalle, $a<b$; $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ continues

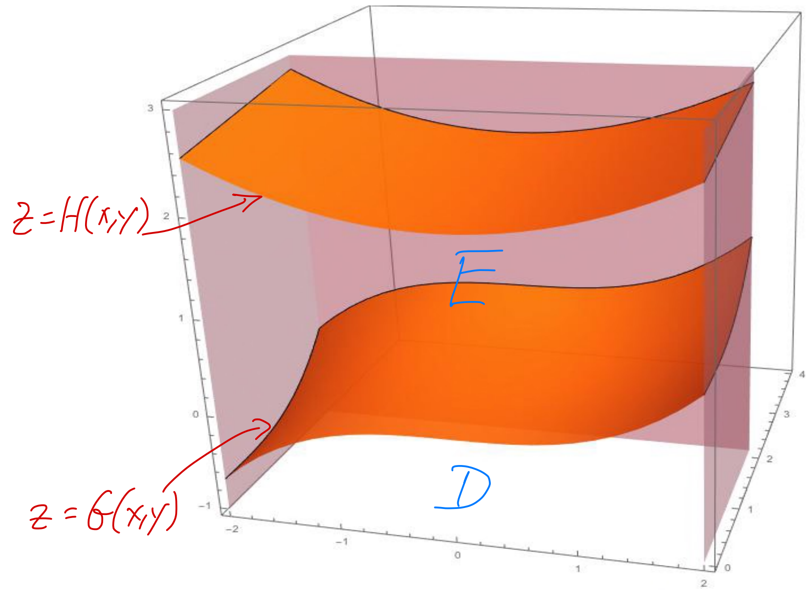
$\varphi_1(x) < \varphi_2(x) \quad \forall x \in]a,b[$.

$$D = \{(x,y) \in \mathbb{R}^2 : a < x < b,\ \varphi_1(x) < y < \varphi_2(x)\}$$

Soient $G, H : \overline{D} \to \mathbb{R}$ continues, $G(x,y) < H(x,y) \ \forall (x,y) \in D$

$$E = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ G(x,y) < z < H(x,y)\}.$$

Soit $f : \overline{E} \to \mathbb{R}$ continue



Alors f est intégrable sur E et on a.

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} \left(\int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz \right) dy \right) dx$$

Remarque : selon la géométrie du domaine les rôles des variables x, y, z peuvent être échangés

Ex. 3 Soit $E = \{(x,y,z) \in \mathbb{R}^3 : \underline{0 < z < y < x < 1}\}$ $\Rightarrow \iiint\limits_E f(x,y,z)\,dx\,dy\,dz = ?$

$$f(x,y,z) = e^{x^3}$$

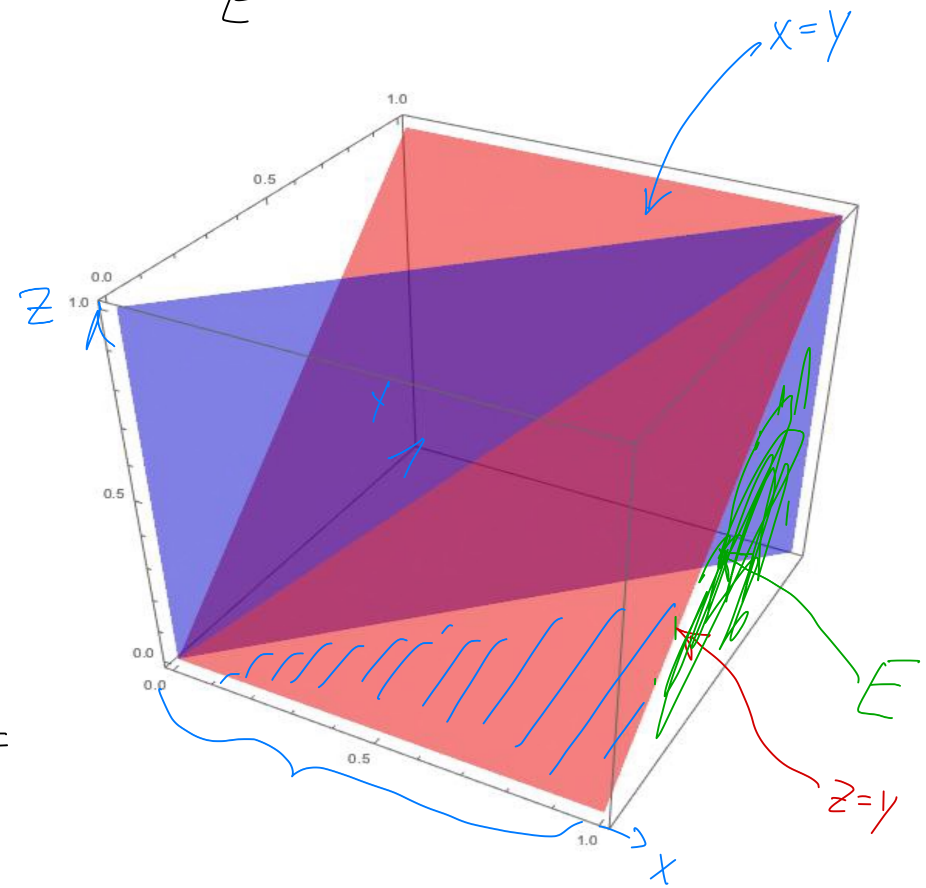
$$E = \{(x,y,z) \in \mathbb{R}^3 : 0 < x < 1,\ 0 < y < x,\ 0 < z < y\}$$

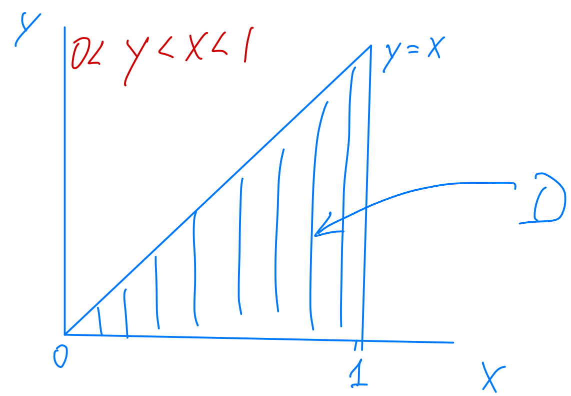
$$\iiint\limits_E f(x,y,z)\,dx\,dy\,dz = \int_0^1 dx \int_0^x dy \int_0^y e^{x^3}\,dz =$$

$$= \int_0^1 dx \int_0^x dy\; z e^{x^3}\Big|_0^y = \int_0^1 dx \int_0^x y e^{x^3}\,dy =$$

$$= \int_0^1 dx \int_0^x y e^{x^3}\,dy = \int_0^1 dx \cdot \frac{1}{2} y^2 e^{x^3}\Big|_0^x = \frac{1}{2}\int_0^1 x^2 e^{x^3}\,dx =$$

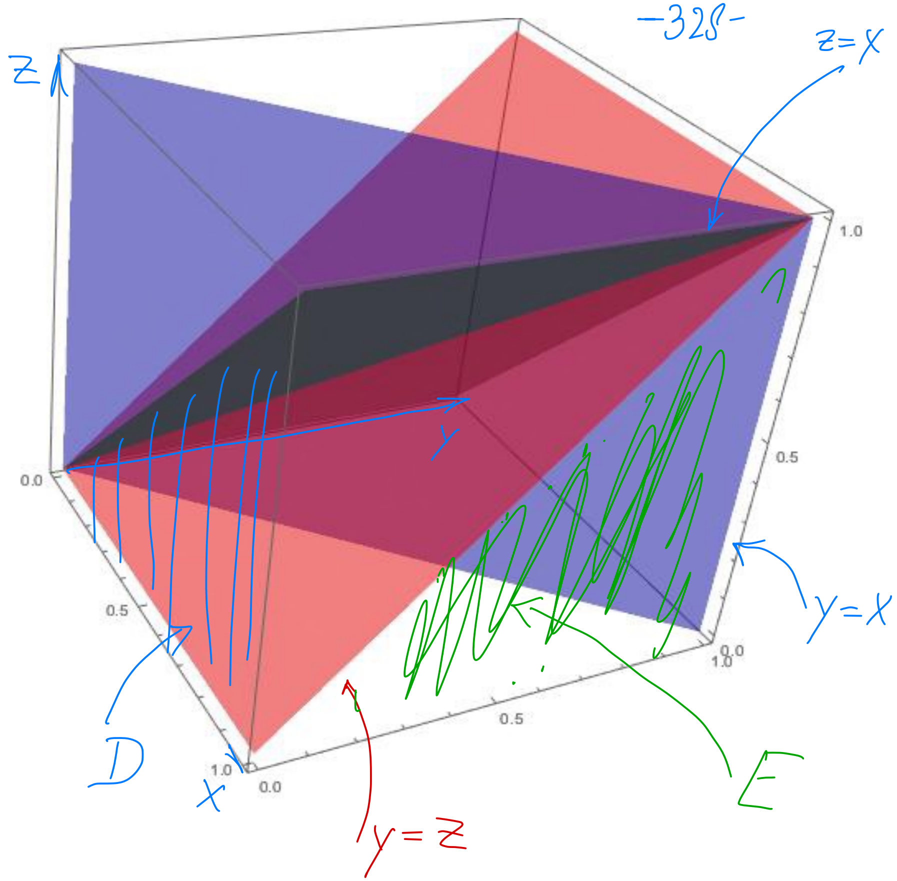
$$= \frac{1}{6}\int_0^1 e^{x^3}\,d(x^3) = \frac{1}{6} e^{x^3}\Big|_0^1 = \underline{\frac{1}{6}(e-1)}.$$





$$E=\{(x,y,z)\in\mathbb{R}^3:\ \underline{0<z<y<x<1}\}$$

$$E=\{(x,y,z)\in\mathbb{R}^3:\ 0<x<1,\ 0<z<x,\ z<y<x\}$$



$$\iiint_E e^{x^3}\,dx\,dy\,dz=\int_0^1 dx\int_0^x dz\int_z^x e^{x^3}dy=\int_0^1 dx\int_0^x dz\left(y e^{x^3}\right)\Big|_z^x=$$

$$=\int_0^1 dx\int_0^x (x-z)e^{x^3}dz=\int_0^1 dx\left(xz-\frac{1}{2}z^2\right)e^{x^3}\Big|_0^x=$$

$$\int_0^1\frac{1}{2}x^2e^{x^3}dx=\frac{1}{6}\int_0^1 e^{x^3}d(x^3)=\frac{1}{6}e^{x^3}\Big|_0^1=$$

$$=\underline{\frac{1}{6}(e-1)}$$

☺

Exercice: calculer le volume du domaine E 6 fois:

$$E=\{(x,y,z)\in\mathbb{R}^3:\ 0<x<1,\ 0<y<x,\ 0<z<y\}$$

$$=\{(x,y,z)\in\mathbb{R}^3:\ 0<x<1,\ 0<z<x,\ z<y<x\}$$

$$\underline{0<z<y<x<1}$$

$$E = \{(x,y,z)\in\mathbb{R}^3 : 0<y<1,\ y<x<1,\ 0<z<y\}$$ Cas 1

$$= \{(x,y,z)\in\mathbb{R}^3 : 0<y<1,\ 0<z<y,\ y<x<1\}$$

$$= \{(x,y,z)\in\mathbb{R}^3 : 0<z<1,\ z<y<1,\ y<x<1\}$$

$$= \{(x,y,z)\in\mathbb{R}^3 : 0<z<1,\ z<x<1,\ z<y<x\}$$ Cas 2

Cas 1 : $$V_E = \iiint_E 1\,dx\,dy\,dz = \int_0^1 dy \int_y^1 dx \int_0^y 1\,dz = \int_0^1 dy \int_y^1 dx \left(z\Big|_{z=0}^{y}\right) = \int_0^1 dy \int_y^1 y\,dx =$$

$$= \int_0^1 dy \left(yx\Big|_{x=y}^{1}\right) = \int_0^1 (y-y^2)\,dy = \frac{1}{2}y^2 - \frac{1}{3}y^3\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}.$$

Cas 2 : $$V_E = \iiint_E 1\,dx\,dy\,dz = \int_0^1 dz \int_z^1 dx \int_z^x 1\cdot dy = \int_0^1 dz \int_z^1 dx \left(y\Big|_{y=z}^{x}\right) = \int_0^1 dz \int_z^1 (x-z)\,dx =$$

$$= \int_0^1 dz \left(\frac{1}{2}x^2 - zx\Big|_{x=z}^{1}\right) = \int_0^1 \left(\frac{1}{2} - z - \frac{1}{2}z^2 + z^2\right) dz = \int_0^1 \left(\frac{1}{2}z^2 - z + \frac{1}{2}\right) dz = \frac{1}{6}z^3 - \frac{1}{2}z^2 + \frac{1}{2}z\Big|_0^1 = \frac{1}{6}$$

D'une façon similaire on trouve pour chaque description du domaine E le même volume $= \frac{1}{6}$.

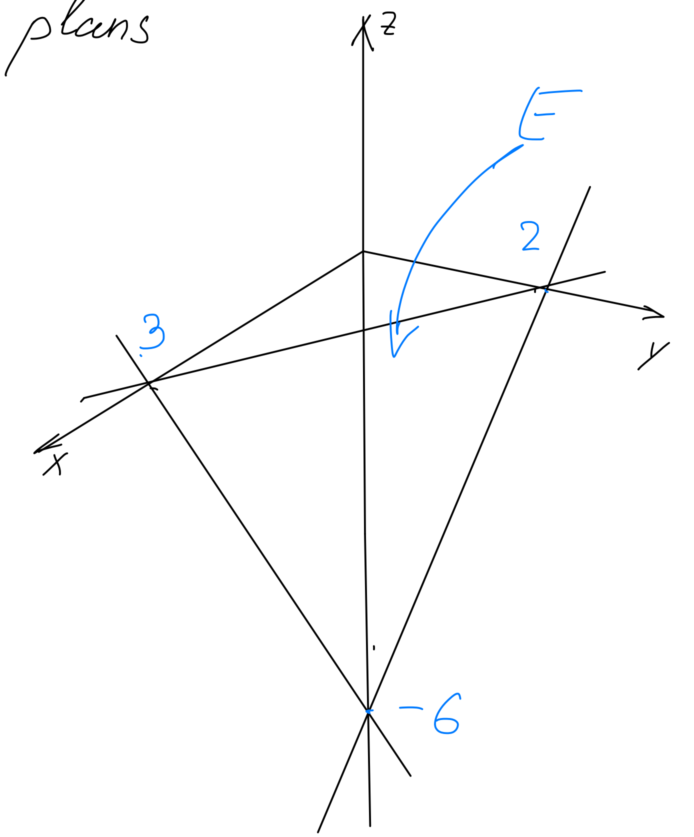
Ex. 4 Trouver le volume de la region $E \subset \mathbb{R}^3$ entre les plans

$$x=0,\ y=0,\ z=0 \text{ et } 2x+3y-z=6$$

$z=0 \Rightarrow 2x+3y=6 \Rightarrow 3y=6-2x \Rightarrow y=2-\frac{2}{3}x$

$x=0, y=0 \Rightarrow z = 2x+3y-6=-6$

$E=\{0<x<3,\ 0<y<2-\frac{2}{3}x,\ 2x+3y-6<z<0\}$

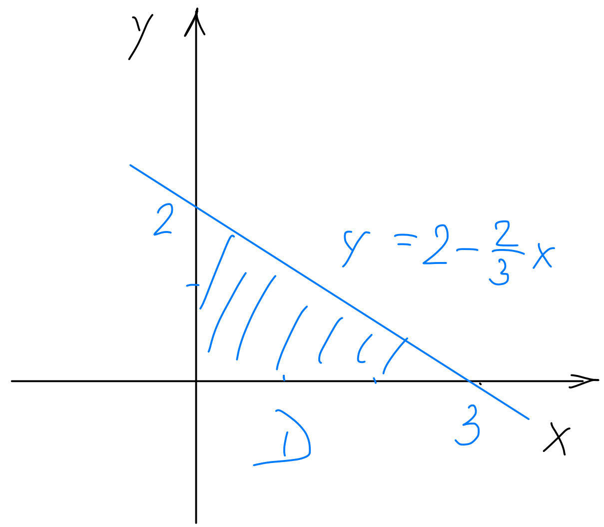


$$V=\iiint_E 1\,dx\,dy\,dz=\int_0^3 dx\int_0^{2-\frac{2}{3}x}dy\int_{2x+3y-6}^{0}dz=\int_0^3 dx\int_0^{2-\frac{2}{3}x}(-2x-3y+6)\,dy=$$

$$=\int_0^3 dx\int_0^{2-\frac{2}{3}x}(-2x-3y+6)\,dy=\int_0^3 dx\left(-2xy-\frac{3}{2}y^2+6y\right)\Big|_0^{2-\frac{2}{3}x}=$$

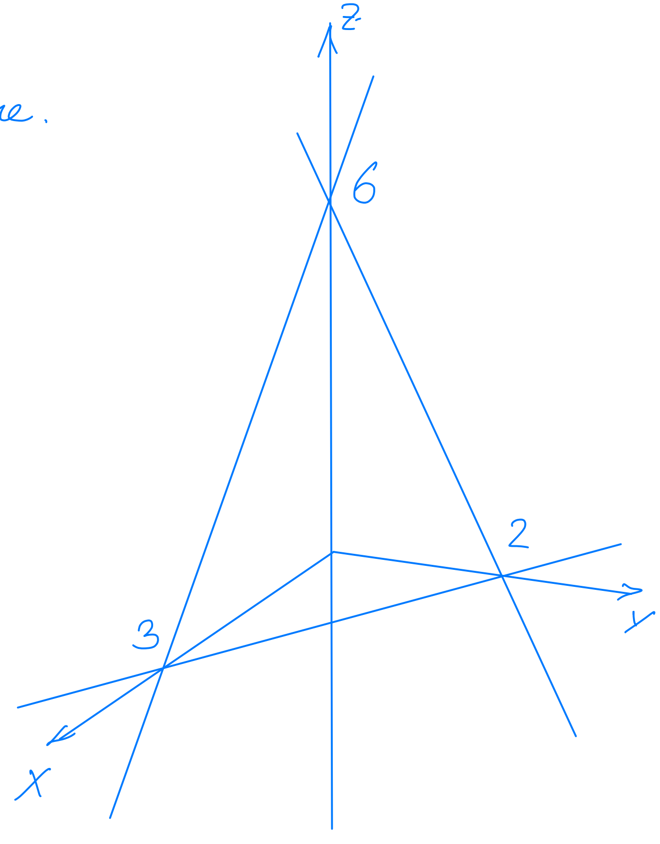
$$=\int_0^3\left(-2x\left(2-\frac{2}{3}x\right)-\frac{3}{2}\underbrace{\left(2-\frac{2}{3}x\right)^2}_{4-\frac{8}{3}x+\frac{4}{9}x^2}+6\left(2-\frac{2}{3}x\right)\right)dx=$$

$$=\int_0^3\left(-4x+\frac{4}{3}x^2-6+4x-\frac{2}{3}x^2+12-4x\right)dx=$$



$$= \int_0^3 \left(-4x + \frac{2}{3}x^2 + 6\right) dx = \left(-2x^2 + \frac{2}{9}x^3 + 6x\right)\Big|_0^3 = (-18 + 6 + 18) = \underline{6}$$

<u>Exercice:</u> Le volume de la région E' entre les plans $x=0$, $y=0$, $z=0$ et $z = 6-2x-3y$ est le même.

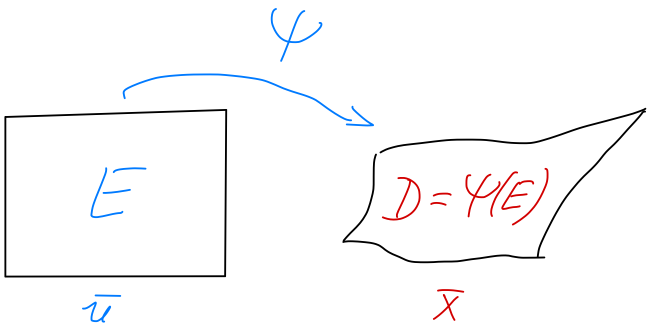


§5.5 Changement de variables dans une intégrale multiple

Thm. Soit $E \subset \mathbb{R}^n$ un sous-ensemble ($\bar{E}$ compact) ; $\Psi : E \to \mathbb{R}^n$ telle que $\Psi \in C^1(E)$ et $\Psi : E \to \Psi(E)$ est bijective ($J_\Psi(\bar{u})$ est inversible $\forall \bar{u} \in E$).

Soit $f : \bar{D} = \overline{\Psi(E)} \to \mathbb{R}$ fonction continue.

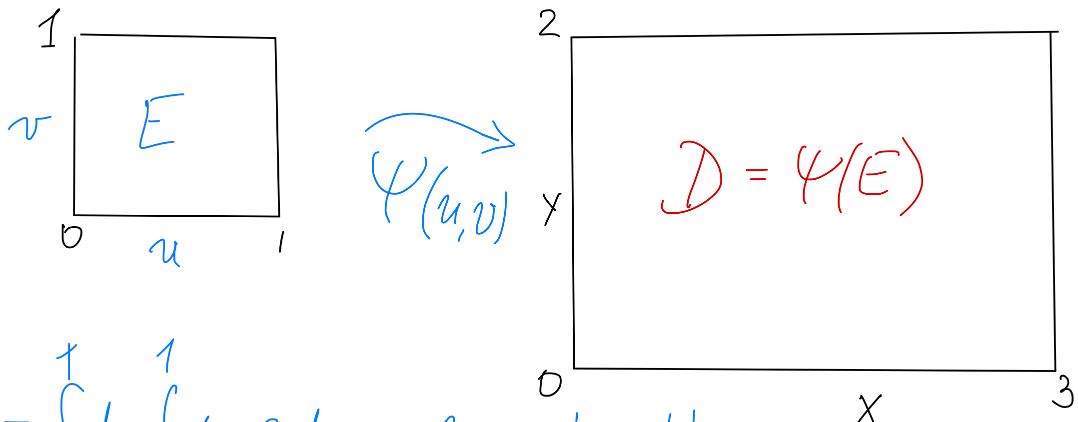
Alors
$$\boxed{\int_D f(\bar{x})\, d\bar{x} = \int_E f(\Psi(\bar{u}))\, |\det J_\Psi(\bar{u})|\, d\bar{u}}$$



Ex. 0 $f(x,y) = 1 = f(u,v)$

$\Psi(u,v) = \begin{pmatrix} 3u \\ 2v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ de classe C^1, bijective

$J_\Psi(u,v) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow |\det J_\Psi(u,v)| = 6$



$$\iint_D f(x,y)\, dx\, dy = \iint_E f(\Psi(u,v)) \cdot |\det J_\Psi(u,v)|\, du\, dv = \int_0^1 du \int_0^1 1 \cdot 6\, dv = 6 \cdot u\Big|_0^1 \cdot v\Big|_0^1 = 6$$

$$\| \quad \int_0^3 dx \int_0^2 1\, dy = 3 \ 2 = 6$$

Ex. 1 $f(x,y)=x^2$, $D=\{(x,y)\in\mathbb{R}^2:\ 0<x<1,\ -x<y<x\}\cup\{(x,y)\in\mathbb{R}^2:\ 1<x<2\ \ x-2<y<2-x\}$.

$$\iint_D f(x,y)\,dx\,dy=\int_0^1 dx\int_{-x}^{x}x^2\,dy+\int_1^2 dx\int_{x-2}^{2-x}x^2\,dy$$

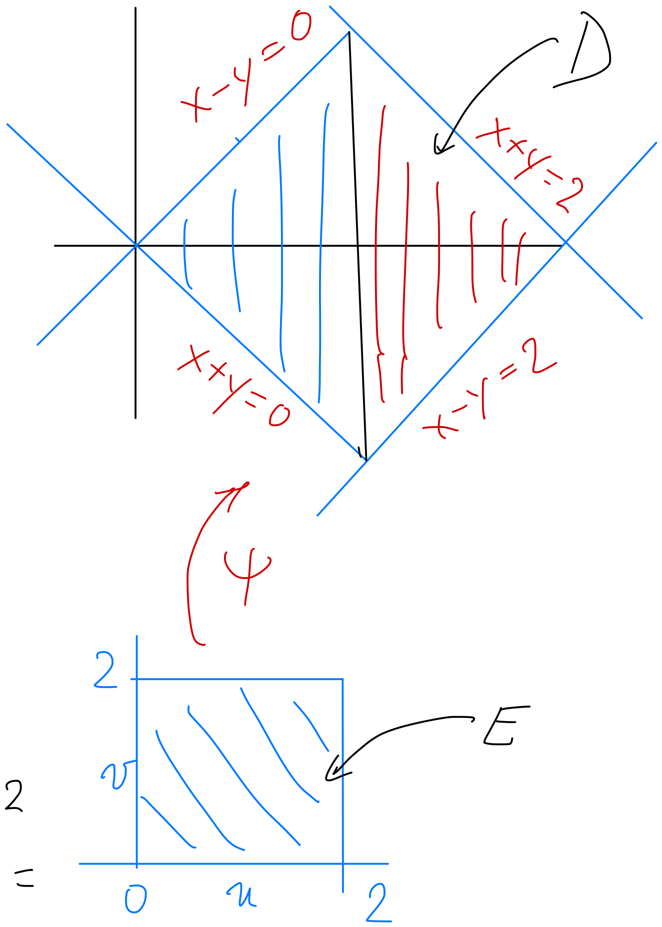
Autrement: $\begin{cases}u=x-y\\ v=x+y\end{cases}\Rightarrow E=\left\{(u,v):\ \begin{matrix}0<u<2\\ 0<v<2\end{matrix}\right\}=\begin{cases}x=\frac{1}{2}(u+v)\\ y=\frac{1}{2}(v-u)\end{cases}$

$$\Rightarrow J_{\psi(u,v)}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}\end{pmatrix}\Rightarrow \det J_{\psi(u,v)}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}\neq 0$$

$$\Rightarrow \iint_D x^2\,dx\,dy=\iint_E\left(\frac{1}{2}(u+v)\right)^2\cdot\underbrace{\frac{1}{2}}_{|\det J_{\psi(u,v)}|}\,du\,dv=\int_0^2 dv\int_0^2\left(\frac{1}{8}(u+v)^2\right)du=$$

$$=\int_0^2 dv\left(\frac{1}{24}(u+v)^3\Big|_{u=0}^{2}\right)=\int_0^2\frac{1}{24}\left((v+2)^3-v^3\right)dv=\frac{1}{24}\,\frac{1}{4}\left((v+2)^4-v^4\right)\Big|_0^2=$$

$$=\frac{1}{24}\cdot\frac{1}{4}\left(4^4-2^4-2^4-0\right)=\frac{1}{6}\cdot\frac{1}{16}\left(16\left(2^4-2\right)\right)=\frac{1}{6}\cdot 14=\underline{\frac{7}{3}}$$



Exercice: Calculer directement : $\iint_D f(x,y)\,dx\,dy = \int_0^1 dx \int_{-x}^{x} x^2\,dy + \int_1^2 dx \int_{x-2}^{2-x} x^2\,dy = \ldots \frac{7}{3}$

Question 21

L'intégrale qui donne le volume d'une sphère solide de rayon R est

A. $$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz$$

~~B.~~ $$\int_{-R}^{R} dy \int_{-\sqrt{R^2-x^2-z^2}}^{\sqrt{R^2-x^2-z^2}} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dz$$

C. $$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz$$

~~D.~~ $$\int_{-R}^{R} dy \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dx \int_{-\sqrt{R^2-x^2-z^2}}^{\sqrt{R^2-x^2-z^2}} dz$$

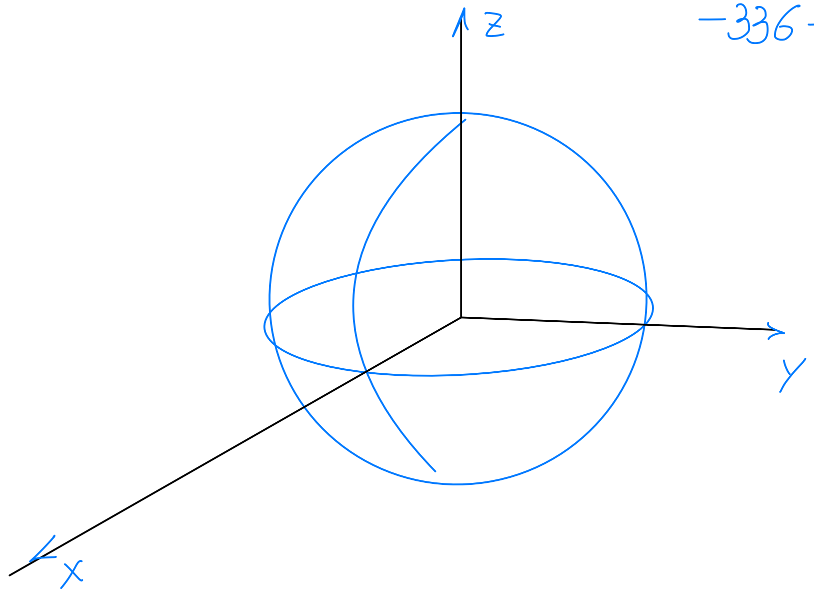
E. Toutes les réponses.

Equation d'une sphère de rayon :

$$x^2+y^2+z^2=R^2$$

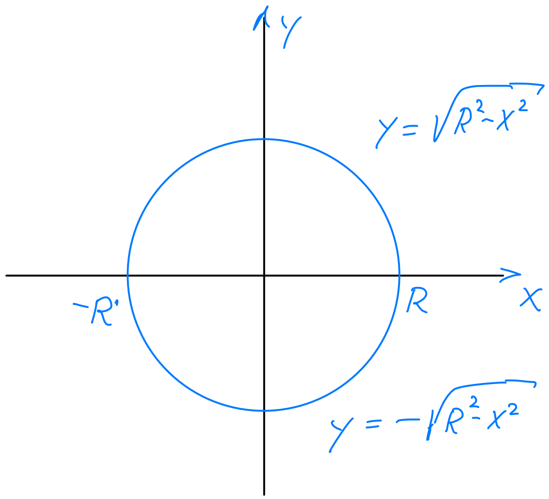
Si $z=0 \Rightarrow x^2+y^2=R^2 \Rightarrow y=\pm\sqrt{R^2-x^2}$

$$D=\{(x,y)\in\mathbb{R}^2 : -R<x<R,\ -\sqrt{R^2-x^2}<y<\sqrt{R^2-x^2}\}$$



$$x^2+y^2+z^2=R^2 \Rightarrow z=\pm\sqrt{R^2-x^2-y^2}$$

$$S=\{(x,y,z)\in\mathbb{R}^3 : -R<x<R,\ -\sqrt{R^2-x^2}<y<\sqrt{R^2-x^2},\ -\sqrt{R^2-x^2-y^2}<z<\sqrt{R^2-x^2-y^2}\}$$

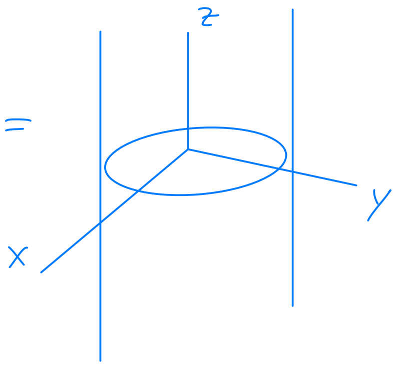


$$V_S=\int_{-R}^{R}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}dy\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}}dz \Rightarrow C$$

Question 21 (B): Quel est l'objet géométrique dont le volume est donné par l'intégrale

A. $\displaystyle\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz$

Intersection de deux cylindres =



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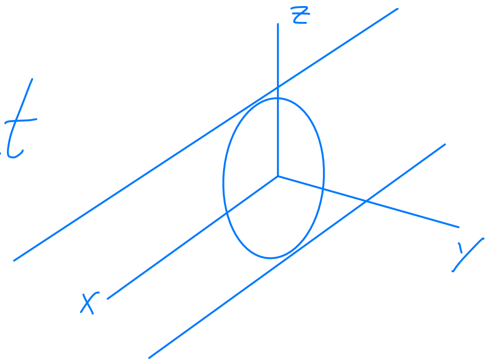

= solide de Steinmetz

Exercice*: calculer le volume de l'intersection des deux cylindres par l'intégrale donnée. $\left(= \frac{16}{3} R^3\right)$.