Test blanc : Lundi le 28 avril

pendant le cours

10h15 → 11h10 test

11h15 → 12h00 correction

Rappel:
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\bar{g}"] \arrow[rr, bend right, "\bar{f}\circ\bar{g}"'] & \mathbb{R}^p \arrow[r, "\bar{f}"] & \mathbb{R}^q \end{tikzcd} \Rightarrow \boxed{J_{\bar{f}\circ\bar{g}(\bar{a})} = J_{\bar{f}(\bar{g}(\bar{a}))} \cdot J_{\bar{g}(\bar{a})}.}$$

Thm. Soient $g, h : I \xrightarrow{C\mathbb{R}} \mathbb{R}$ fonctions continûment dérivables, et $f : \overset{x}{J} \times \overset{t}{I} \to \mathbb{R}$ telle que $\frac{\partial f}{\partial t}$ est continue sur I.

ouverts

Alors
$$\boxed{F(t) = \int_{h(t)}^{g(t)} f(x,t)\,dx}$$
est continûment dérivable sur I, et on a :

$$\boxed{F'(t) = f(g(t),t)\cdot g'(t) - f(h(t),t)\cdot h'(t) + \int_{h(t)}^{g(t)} \frac{\partial f}{\partial t}(x,t)\,dx}$$

Ex. $\displaystyle\int_0^1 \frac{x-1}{\ln x}\,dx = ?$ $\displaystyle\int \frac{x-1}{\ln x}\,dx$ ne s'exprime pas en fonctions élémentaires.

$$\lim_{x\to 0}\frac{x-1}{\ln x} = 0\ ,\quad \lim_{x\to 1}\frac{x-1}{\ln x} = \lim_{y\to 0}\frac{y}{\ln(1+y)} = 1$$

$$\rightsquigarrow \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx \overset{\alpha>-1}{=} I(\alpha) \Rightarrow I'(\alpha) = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^\alpha-1}{\ln x}\right)dx = \int_0^1 \frac{x^\alpha\cdot\cancel{\ln x}}{\cancel{\ln x}}\,dx =$$

$$\underset{\alpha>-1}{\int_0^1 x^\alpha dx} = \frac{1}{\alpha+1}\,x^{\alpha+1}\Big|_0^1 = \frac{1}{\alpha+1} \Rightarrow I'(\alpha) = \frac{1}{\alpha+1}$$

$$\Rightarrow I(\alpha) = \ln(\alpha+1) + C\ ;\quad \text{On sait que } I(0) = \int_0^1 \frac{x^0-1}{\ln x}\,dx = \int_0^1 0\,dx = 0.$$

$$\Rightarrow I(0) = \ln(0+1) + C = 0 = C \Rightarrow C = 0$$

$$\Rightarrow \underset{\alpha>-1}{I(\alpha) = \ln(\alpha+1)} \Rightarrow I(1) = \int_0^1 \frac{x-1}{\ln x}\,dx = \ln(1+1) = \ln 2$$

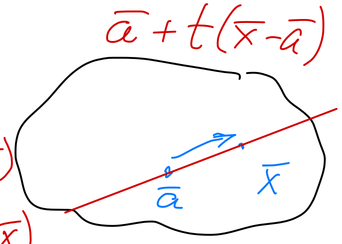
§4.8 Formule de Taylor.

Thm. Soit $f: E \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}$ de classe C^{p+1} au voisinage de $\bar{a} \in E$.
Alors il existe $\delta > 0$ tel que pour tout $x \in B(\bar{a}, \delta) \cap E$ il existe $0 < \theta < 1$ tel que

$$f(\bar{x}) = F(0) + F'(0) + \frac{1}{2} F''(0) + \ldots + \frac{1}{p!} F^{(p)}(0) + \frac{1}{(p+1)!} F^{(p+1)}(\theta)$$

où $F : \underset{\supset [0,1]}{I} \to \mathbb{R}$, $\quad F(t) = f(\bar{a} + t(\bar{x} - \bar{a}))$

$F(0) = f(\bar{a})$
$F(1) = f(\bar{x})$



Explication: $f(\bar{x}) = F(1)$, $f(\bar{a}) = F(0)$; $F(t)$ de classe C^{p+1} sur I

Analyse I $\Rightarrow$ la formule de Taylor pour $F(t)$, fonction d'une seule variable

$$\Rightarrow F(t) = F(0) + F'(0) \cdot t + \frac{1}{2} F''(0) \cdot t^2 + \ldots + \frac{1}{p!} F^{(p)}(0) \cdot t^p + \frac{1}{(p+1)!} F^{(p+1)}(\theta) t^{p+1}, \quad \theta \text{ est entre } 0 \text{ et } t$$

$\downarrow t = 1$

$$\Rightarrow f(\bar{x}) = \underbrace{F(0)}_{= f(a)} + F'(0) + \frac{1}{2} F''(0) + \ldots + \frac{1}{p!} F^{(p)}(0) + \frac{1}{(p+1)!} F^{(p+1)}(\theta) \qquad 0 < \theta < 1$$

▨

Remarque. $$F'(0) = \lim_{t \to 0} \frac{f(\bar{a} + t(\bar{x} - \bar{a})) - f(\bar{a})}{t} = \underbrace{Df(\bar{a}, (\bar{x} - \bar{a}))}_{\text{dérivée directionnelle}} = \langle \nabla f(\bar{a}), \bar{x} - \bar{a} \rangle$$

$$\boxed{f(\bar{x}) = F(0) + F'(0) + \frac{1}{2} F''(0) + \ldots + \frac{1}{p!} F^{(p)}(0) + \text{le reste}}$$

le polynôme de Taylor de f d'ordre p au point $\bar{a}$.

En particulier, Cas $n=2$. $\bar{a}=(a,b)$; $\bar{x}=(x,y)$. f de classe C^{p+1}, $p\geq 2$.

$f(x,y): E\to\mathbb{R}$, on cherche le polynôme de Taylor d'ordre p autour de (a,b).

$F(t)=f(a+t(x-a),\ b+t(y-b))$. Trouver $F'(t)$, $F''(t)$ en termes de f.

$F(t)=f\circ g(t)$, $f:\mathbb{R}^2\to\mathbb{R}$ $g:\mathbb{R}\to\mathbb{R}^2$, $g(t)=(a+t(x-a)\ b+t(y-b))^T$

$$\Rightarrow F'(t)=J_F(t)=J_f(g(t))\cdot J_g(t)=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{pmatrix}\begin{pmatrix}g_1'(t)\\ g_2'(t)\end{pmatrix}=\frac{\partial f}{\partial x}(x-a)+\frac{\partial f}{\partial y}(y-b).$$

$$\Rightarrow F'(0)=\frac{\partial f}{\partial x}(a,b)\cdot(x-a)+\frac{\partial f}{\partial y}(a,b)\cdot(y-b)$$

$F''(t)=?$ $\quad F'(t)=\underbrace{\frac{\partial f}{\partial x}g_1'(t)}+\underbrace{\frac{\partial f}{\partial y}\cdot g_2'(t)}$

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\cdot g_1'(t)\right)=\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)g_1'(t)}+\frac{\partial f}{\partial x}\cdot \overbrace{g_1''(t)}^{0}=$$

$$=\left(\frac{\partial^2 f}{\partial x^2}\cdot g_1'(t)+\frac{\partial^2 f}{\partial y\partial x}\cdot g_2'(t)\right)g_1'(t)$$

$$F''(t)=\overbrace{\frac{\partial^2 f}{\partial x^2}(g_1'(t))^2+\frac{\partial^2 f}{\partial y\partial x}(g_1'(t)g_2'(t))}+\overbrace{\frac{\partial^2 f}{\partial x\partial y}(g_2'(t)g_1'(t))+\frac{\partial^2 f}{\partial y^2}(g_2'(t))^2}=$$

Thm de Schwartz f de classe C^p, $p\geq 3$

$$F''(0)=\frac{\partial^2 f}{\partial x^2}(x-a)^2+2\frac{\partial^2 f}{\partial x\partial y}(x-a)(y-b)+\frac{\partial^2 f}{\partial y^2}(y-b)^2.$$

$g_1(t)=a+t(x-a)$
$g_1'(t)=x-a$
$g_1''(t)=0$
$g_2(t)=b+t(y-b)$
$g_2'(t)=y-b$
$g_2''(t)=0$

$$\boxed{F''(0) = \underset{1}{\frac{\partial^2 f}{\partial x^2}(a,b)\cdot(x-a)^2} + \underset{2}{2\frac{\partial^2 f}{\partial x \partial x}(a,b)\cdot(x-a)(y-b)} + \underset{1}{\frac{\partial^2 f}{\partial y^2}(a,b)\cdot(y-b)^2}}$$

D'une façon similaire, on obtient:

$$F^{(3)}(0) = \underset{1}{\frac{\partial^3 f}{\partial x^3}\cdot(x-a)^3} + \underset{3}{3\frac{\partial^3 f}{\partial x^2 \partial y}\cdot(x-a)^2(y-b)} + \underset{3}{3\frac{\partial^3 f}{\partial x \partial y^2}\cdot(x-a)(y-b)^2} + \underset{1}{\frac{\partial^3 f}{\partial y^3}\cdot(y-b)^3}$$

Les coefficients binomiaux $C_p^k = \frac{p!}{k!(p-k)!}$

$$\Rightarrow F^{(p)}(0) = \sum_{k=0}^{p} \frac{\partial^p f}{\partial x^k \partial y^{p-k}} C_p^k (x-a)^k (y-b)^{p-k}$$

Souvent on utilise l'approximation de Taylor d'ordre 2:

$$f(x,y) = \underbrace{f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)\cdot(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)\cdot(y-b)^2\right)}_{P_2 f(a,b) \text{ - polynôme de Taylor de } f \text{ d'ordre 2 autour de } (a,b).}$$

$$+ \underbrace{\varepsilon\left(\|(x,y)-(a,b)\|^2\right)}_{\text{le reste}}$$

Ex 1 $f(x,y) = y\cos x + x\sin y$. Trouver le polynôme de Taylor d'ordre 2 de $f(x,y)$ autour de $(\pi, 0)$.

$$P_2 f_{(\pi,0)} = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)\cdot(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)\cdot(y-b)^2\right)$$

$$\frac{\partial f}{\partial x} = -y\sin x + \sin y\Big|_{(\pi,0)} = 0 \ ; \quad \frac{\partial f}{\partial y} = \cos x + x\cos y\Big|_{(\pi,0)} = -1+\pi$$

$$\frac{\partial^2 f}{\partial x^2} = -y\cos x\Big|_{(\pi,0)} = 0 \ ; \quad \frac{\partial^2 f}{\partial y^2} = -x\sin y\Big|_{(\pi,0)} = 0 \quad \frac{\partial^2 f}{\partial x \partial y} = -\sin x + \cos y\Big|_{(\pi,0)} = 1.$$

$$\Rightarrow P_2 f_{(\pi,0)} = 0 + 0(x-\pi) + (\pi-1)(y-0) + \frac{1}{2}\big(2\cdot 1(x-\pi)(y-0)\big) = \underline{(\pi-1)\cdot y + (x-\pi)\cdot y}$$

$$= -y + xy.$$

Autrement. Soit $s = \underbrace{x-\pi}_{\text{petit}} \Rightarrow x = s+\pi \Rightarrow$

$$f(x,y) = y\cos x + x\sin y = y\cos(\pi+s) + (\pi+s)\sin y = y(\underbrace{\cos\pi}_{=-1}\cos s - \underbrace{\sin\pi}_{=0}\sin s) + (\pi+s)\sin y =$$

$$-y(\cos s) + (\pi+s)\sin y \overset{DL}{\approx} -y\left(1-\frac{s^2}{2}\right) + (\pi+s)\cdot y = -y + \underbrace{(\pi+s)}_{x}y = \underline{-y + \pi y + (x-\pi)y}$$

même qu'avant

Ex 2 $f(x,y) = \cos(e^{x+y}+x-y)$. Trouver le polynôme de Taylor d'ordre 2 de $f(x,y)$ autour de $(x,y)=(0,0)$.

Par les DL (x,y) autour de $(0,0)$

Notation: $\varepsilon(x,y)^p$: $\lim \frac{\varepsilon(x,x)^p}{\|(x,y)\|^p} = 0$

$$\cos(e^{x+y}+x-y) = \cos\left(1+x+y+\frac{1}{2}(x+y)^2+\varepsilon(x,y)^2+x-y\right) =$$

$$= \cos\left(1+2x+\frac{1}{2}(x+y)^2+\varepsilon(x,y)^2\right) = \cos 1 \cos\left(2x+\frac{1}{2}(x+y)^2+\varepsilon(x,y)^2\right) - \sin 1 \sin\left(2x+\frac{1}{2}(x+y)^2+\varepsilon(x,y)^2\right)$$

$$= \cos 1\left(1-\frac{1}{2}(2x)^2\right)+\varepsilon(x,y)^2 - \sin 1\left(2x+\frac{1}{2}(x+y)^2\right)$$

$$\Rightarrow P_2 f(0,0) = \cos 1 - 2\cos 1\cdot x^2 - 2\sin 1\cdot x - \frac{1}{2}\sin 1\,(x^2+2xy+y^2) =$$

$$= \underbrace{\cos 1 - 2\sin 1\cdot x - \left(2\cos 1+\frac{1}{2}\sin 1\right)x^2 - \frac{1}{2}\sin 1\cdot y^2 - \sin 1\, xy}_{P_2 f(0,0)}$$

Exercice: Obtenir le même résultat par la formule de Taylor avec les dérivées partielles.

Ex 3 e^{xy} Polynôme de Taylor d'ordre 2 autour de $(-1,2)$.

$x=-1+t,\quad y=2+s \qquad t,s$ petits.

$$\tilde{f}(t,s) = e^{(-1+t)(2+s)} = e^{-2}e^{\underbrace{-s+2t+ts}_{\text{petit}}} \Rightarrow \text{DL pour } e$$

$$\tilde{f}(t,s) = e^{-2}\left(1+(-s+2t+ts)+\frac{1}{2}(-s+2t+ts)^2\right)+\varepsilon(t,s)^2$$

$$P_2 f(-1,2) = e^{-2}\left(1-s+2t+\underline{ts}+\frac{1}{2}s^2+2t^2-\underline{2st}\right) =$$

$$= e^{-2}+e^{-2}\left(-(y-2)+2(x+1)+\frac{1}{2}(y-2)^2+2(x+1)^2-(x+1)(y-2)\right).$$

$$\mathbb{R} \xrightarrow{\bar{g}} \mathbb{R}^3 \xrightarrow{f} \mathbb{R} \quad (F = f\circ\bar{g}) \Rightarrow F'(t) = J_{f\circ\bar{g}}(t) = J_{f(\bar{g}(t))}\cdot J_{\bar{g}(t)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{pmatrix}\begin{pmatrix} g_1'(t) \\ g_2'(t) \\ g_3'(t)\end{pmatrix}$$

Cas $n=3$. $f(x,y,z)$ autour de $(a,b,c) \in E \subset \mathbb{R}^3$, fonction de classe C^3

$$g(t) = \begin{pmatrix} a+t(x-a) \\ b+t(y-b) \\ c+t(z-c)\end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t)\end{pmatrix} \; ; \; F'(t) = \frac{\partial f}{\partial x}\cdot\frac{\partial g_1}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial g_2}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial g_3}{\partial t} \; ; \; F = f\circ g$$

$$F(t) = f(g_1(t), g_2(t), g_3(t))$$

$$\Rightarrow F'(0) = \frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)\cdot(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c) \; , \ldots\ldots \quad F''(0).$$

Exercice: écrire le polynôme de Taylor de $f(x,y,z)$ d'ordre 2 autour de (a,b,c)

$$f(x,y,z) = \underbrace{f(a,b,c) + \langle \nabla f(a,b,c), (x-a, y-b, z-c)\rangle}_{P_1 f(a,b,c)} + \ldots$$

$$\underbrace{+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b,c)\cdot(x-a)^2 + \frac{\partial^2 f}{\partial y^2}(a,b,c)\cdot(y-b)^2 + \frac{\partial^2 f}{\partial z^2}(a,b,c)(z-c)^2 + 2\frac{\partial^2 f}{\partial x\partial y}(x-a)(y-b) + 2\frac{\partial^2 f}{\partial y\partial z}(y-b)(z-c) + 2\frac{\partial^2 f}{\partial x\partial z}(x-a)(z-c)\right)}_{P_2 f(a,b,c)} +$$

$$+ \varepsilon\left(\|(x,y,z)-(a,b,c)\|^2\right)$$

2 Méthodes. (1) La formule de Taylor en plusieurs variables.

(2) Utiliser les développements limités connus d'Analyse 1.

Ex 4 $\quad f(x,y,z) = \dfrac{\cos(y+2z)}{1-x} \quad x \neq 1, \quad$ Trouver $P_2 f(0,0,0)$

Méthode DL : x, y, z sont petits autour de $(0,0,0)$

$$\frac{1}{1-x} = 1 + x + x^2 + \mathcal{E}(x^2) \quad ; \quad \cos(\underbrace{y+2z}_{u \text{ petit}}) = \cos(u) = 1 - \frac{1}{2}u^2 + \mathcal{E}(u^2) = 1 - \frac{1}{2}(y+2z)^2 + \mathcal{E}^2(y,z)$$

$$P_2 f(0,0,0) = (1+x+x^2)\left(1 - \frac{1}{2}(y+2z)^2\right) = 1 + x + x^2 - \frac{1}{2}(y^2 + 4yz + 4z^2) =$$

$$1 + x + x^2 - \frac{1}{2}y^2 - 2yz - 2z^2$$

coeff

$$\frac{\partial f}{\partial x} = -\frac{(-1)}{(1-x)^2}\cos(y+2z)\Big|_{(000)} = 1 \qquad \frac{\partial f}{\partial y} = \frac{-\sin(y+2z)}{1-x}\Big|_{(000)} = 0 \qquad \frac{\partial f}{\partial z} = \frac{-2\sin(y+2z)}{1-x}\Big|_{(000)} = 0$$

Exercice : Vérifier la formule obtenue par le calcul de dérivées partielles

Méthode 1. directe mais fastidieuse : calculer les dérivées partielles $\rightsquigarrow$ formule de Taylor multivariable

Méthode 2 efficace mais nécessite un traitement soigneux et la maitrise de DL.

Question 15

$$f(x,y) = \frac{1}{\sin(x+y)}$$

Le coefficient de $\left(x-\frac{\pi}{2}\right)^2 y^2$ dans le polynôme de Taylor d'ordre 4 autour de $(x,y) = \left(\frac{\pi}{2}, 0\right)$ est

A. $\frac{1}{24}$

B. $\frac{5}{4}$

C. $\frac{5}{24}$

D. $\frac{5}{6}$

$$f(x) = \frac{1}{\sin(x+y)}$$

Le coefficient de $(x-\frac{\pi}{2})^2 y^2$ dans le polynôme de Taylor d'ordre 4 autour de $(x,y) = (\frac{\pi}{2}, 0)$ est

$P_4 f(\frac{\pi}{2}, 0)$.

$x = \frac{\pi}{2} + t \Rightarrow t = x - \frac{\pi}{2}$ petif.

$$f(x) = \frac{1}{\sin(\frac{\pi}{2} + \underbrace{t+y}_{s \text{ petit}})} = \frac{1}{\sin(\frac{\pi}{2}+s)} = \frac{1}{\underbrace{\sin\frac{\pi}{2}}_{=1}\cos s + \underbrace{\cos\frac{\pi}{2}}_{=0}\sin s} = \frac{1}{\cos s} = \frac{1}{1-\frac{s^2}{2}+\frac{s^4}{4!}+\varepsilon(s^4)} =$$

$$= \frac{1}{1-\left(\frac{s^2}{2}-\frac{s^4}{4!}+\varepsilon(s^4)\right)}$$

$$\Rightarrow P_4 f = 1 + \left(\frac{s^2}{2} - \frac{s^4}{4!}\right) + \left(\frac{s^2}{2} - \cancel{\frac{s^4}{4!}}\right)^2 = 1 + \frac{s^2}{2} - \frac{s^4}{4!} + \frac{s^4}{4} = 1 + \frac{s^2}{2} + \frac{-1+6}{24}s^4 = 1 + \frac{s^2}{2} + \frac{5}{24}s^4.$$

$$s^4 = \left((x-\tfrac{\pi}{2})+y\right)^4 = (x-\tfrac{\pi}{2})^4 + 4(x-\tfrac{\pi}{2})^3 y + \underline{\underline{6(x-\tfrac{\pi}{2})^2 y^2}} + 4(x-\tfrac{\pi}{2})y^3 + y^4$$

$\Rightarrow$ le coefficient de $(x-\frac{\pi}{2})^2 y^2$ dans $P_4 f(\frac{\pi}{2}, 0)$ est $6 \cdot \frac{5}{24} = \boxed{\frac{5}{4}}$. $\Rightarrow B$

Le Laplacien d'une fonction de classe C^2.

Déf. Soit $f : E \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}$ de classe C^2 sur E. La fonction $\Delta f : E \to \mathbb{R}$

$$\boxed{\Delta f(x_1, \dots x_n) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_n^2}}$$

est le Laplacien de f.

Ex 1. $f(x,y) = e^{x+y} + 2x^2y$

$$\frac{\partial f}{\partial x} = e^{x+y} + 4xy \qquad \frac{\partial f}{\partial y} = e^{x+y} + 2x^2$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x+y} + 4y \qquad \frac{\partial^2 f}{\partial y^2} = e^{x+y}$$

$$\Rightarrow \Delta f(x,y) = 2e^{x+y} + 4y.$$

Exercice : Exprimer $\Delta f(x,y)$ en coordonnées polaires (voir cours 17, p. 200 et Série 9).

Déf. Une fonction telle que $\Delta f = 0$ sur $E \subset \mathbb{R}^2$ s'appelle harmonique.

Ex 3. $f(x,y) = x^2 - y^2 : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y$$

$$\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 - 2 = 0$$

f est harmonique sur $\mathbb{R}^2$

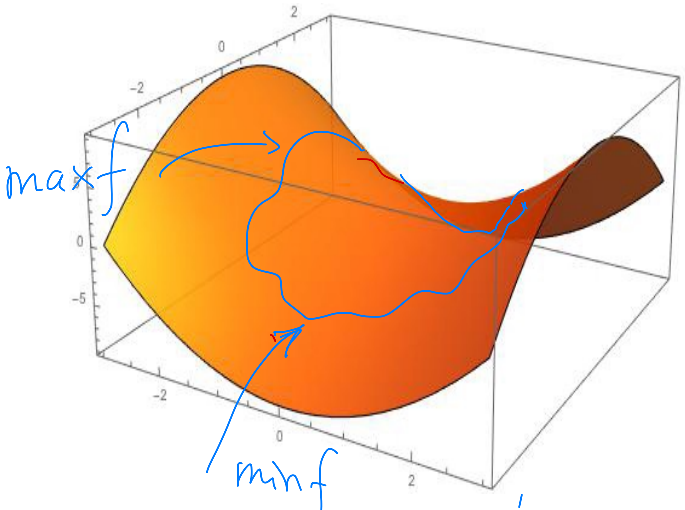
$g(x,y) = x^2 + y^2 : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 2y$$

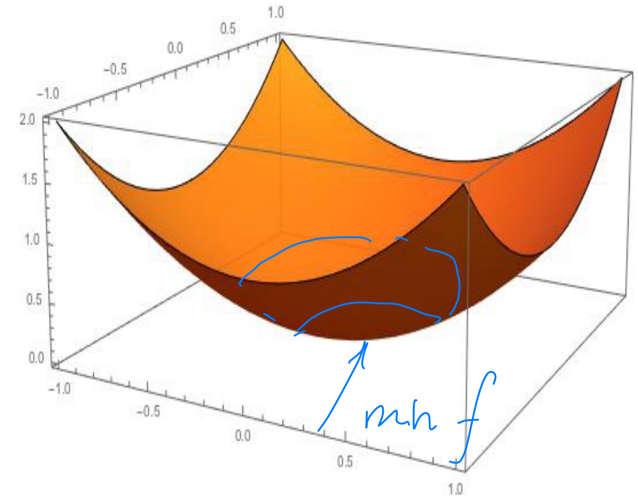
$$\Delta g(x,y) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 2 + 2 = 4$$

g n'est pas harmonique

Une fonction harmonique sur un domaine compact atteint son min et son max sur la frontière du domaine. (Sans démonstration)



fonction harmonique.



n'est pas harmonique