Analysis I (English)
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Analysis 1 - Key to open questions from previous mock exams

1. See Fall 2018

Solution:

(a) We prove the claim by induction. First, we prove by induction that $x_n \geq 1$ for every n. The base case $n = 0$ is given. Now, we use the strong form of induction. Fix $n \geq 0$, and assume that $x_k \geq 1$ for every $0 \leq k \leq n$. Then, if $n+1 = 1$ (i.e., $n = 0$), we know that $x_n \geq 1$. So, assume that we have $n \geq 1$ (so that $n - 1$ makes sense in $\mathbb{N}$). Then, we have

$$x_{n+1} = x_n + x_{n-1} \geq 1 + 1 = 2 \geq 1.$$

So, we settled the inductive step

Now, we turn to proving the main claim. It is given for $n = 0$ and $n = 1$. So, we set an inductive argument starting from $n = 2$. For the base case, we have

$$x_2 = x_1 + x_0 = 1 + 1 = 2 \geq 2,$$

so the base case is settled. Now, fix $n \geq 2$, and assume that $x_n \geq n$. Then, we have

$$x_{n+1} = x_n + x_{n-1} \geq n + x_{n-1} \geq n + 1,$$

where we use the inductive hypothesis in the first inequality, and we use that $x_k \geq 1$ for every $k \geq 0$ in the second inequality. So, the claim is settled.

Note: if one wants to avoid strong induction to prove that $x_k \geq 1$ for every $k \geq 0$, we can use a proof by contradiction as follows.

Assume that it is not true that $x_n \geq 1$ for every $n \in \mathbb{N}$. Then, the set $S = \{n \in \mathbb{N} | x_n < 1\}$ is not empty. Thus, as $S \subseteq \mathbb{N}$, S has a minimum. Let N be its minimum. We know that $N \neq 0$ and $N \neq 1$, as $x_0 = x_1 = 1$. So, we have $N \geq 2$, and by the recursive formula, we may write

$$1 > x_N = x_{N-1} + x_{N-2} \geq 1 + 1 = 2,$$

which is absurd, as $1 > 2$ is false. Here, we used the fact that $N \in S$ to argue that $1 > x_N$. Then, we used that N is the minimum of S to argue that x_{N-1} and x_{N-2} are both ≥ 1.

(b) By part (a), for every n, we have

$$x_n \geq n.$$

Then, for $n \geq 1$, we have

$$\sqrt[x_n]{x_n} \geq \sqrt[x_n]{n} \geq \sqrt[x_n]{1} = 1,$$

where we used that $x_n \geq n \geq 1$ and the fact that $\sqrt[x_n]{x}$ is an increasing function in x. Since $x_n \geq n$ for every n, we have $\lim_{n\to\infty} x_n = +\infty$. Furthermore, (x_n) is strictly increasing for $n \geq 2$. Then, as we know from class that $\lim_{k\to\infty} \sqrt[k]{k} = 1$, by a change of variable, we know that $\lim_{n\to\infty} \sqrt[x_n]{x_n} = 1$. Since both $sqrt[x_n]x_n$ and $\sqrt[x_n]{1}$ converge to 1, we can conclude that $\sqrt[x_n]{n}$ converges to 1 by the squeeze theorem.

2. See Fall 2019

Solution: To make sure that the recursion defines a sequence, we need to make sure that we can take the square root of each step. So, to start, we show that $x_n \geq 1$ for every n, so that then $\frac{3}{2}x_n - \frac{1}{2} \geq 1$, and we can take the root defining x_{n+1}. We proceed by induction. The base case $n = 0$ is given. So now, fix n, and assume that $x_n \geq 1$. Then, we have

$$x_{n+1} = \sqrt{\frac{3}{2}x_n - \frac{1}{2}} \geq \sqrt{1} = 1.$$

(a) We proceed by induction, and our base case $n = 0$ is given by assumption. Thus, we proceed with the inductive step. Fix $n \geq 0$, and assume that $x_n \geq 1$. Then, we have

$$x_{n+1} = \sqrt{\frac{3}{2}x_n - \frac{1}{2}} \geq \sqrt{\frac{3}{2} - \frac{1}{2}} = \sqrt{1} = 1.$$

This settles the inductive step.

(b) Now, we proceed by induction. The base case corresponds to $n = 1$, i.e., we have to show $x_1 \leq x_0$. So, we have

$$x_1 = \sqrt{\frac{3}{2}2 - \frac{1}{2}} = \sqrt{\frac{5}{2}} < \sqrt{\frac{8}{2}} = 2 = x_0.$$

Now, fix $n \geq 1$, and assume that $x_n \leq x_{n-1}$. Then, we have

$$x_{n+1} = \sqrt{\frac{3}{2}x_n - \frac{1}{2}} \leq \sqrt{\frac{3}{2}x_{n-1} - \frac{1}{2}} = x_n,$$

where we used that $x_n \leq x_{n-1}$ and that the function $\sqrt{x}$ is increasing in x.

(c) By part (b), (x_n) is a monotonic sequence. By the preamble, we know that (x_n) is bounded below by 1. Then, a sequence that is decereasing and bounded below is bounded. Then, we can conclude, as bounded monotonic sequences are convergent.

Note 1: if we do not observe that $x_n \geq 1$ for all n, we anyway know that $x_n \geq 0$, as it is a square root (on the other hand, we needed to show $x_n \geq 1$ to show that the recursion is infinite).

Note 2: we can actually compute the limit. The recursion gives us $x = \sqrt{\frac{3}{2}x - \frac{1}{2}}$. Then, we have $x^2 = \frac{3}{2}x - \frac{1}{2}$. This equation has solutions 1 and $\frac{1}{2}$. Since we showed that $x_n \geq 1$ for all n, the limit has to be 1.