Duration : 60 minutes

Analysis I
Midterm test
Autumn 2017

- For the **multiple choice** questions, we give

  +3 points if your answer is correct,

  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.

- For the **true/false** questions, we give

  +1 points if your answer is correct,

  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.

First part: multiple choice questions

For each question, cross the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The set of numbers $z \in \mathbb{C}$ such that $z^3 = \dfrac{(3+3i)^3}{2i+2}$ is

- ☐ $\left\{-3\,\mathrm{i},\ \frac{3}{2}\big(1+\sqrt{3}\,\mathrm{i}\big),\ -\frac{3}{2}\big(1-\sqrt{3}\,\mathrm{i}\big)\right\}$
- ☐ $\left\{-3\,\mathrm{i},\ \frac{3}{2}\big(\sqrt{3}+\mathrm{i}\big),\ -\frac{3}{2}\big(\sqrt{3}-\mathrm{i}\big)\right\}$
- ☐ $\left\{3\,\mathrm{i},\ \frac{3}{2}\big(\sqrt{3}-\mathrm{i}\big),\ -\frac{3}{2}\big(\sqrt{3}+\mathrm{i}\big)\right\}$
- ☐ $\left\{3\,\mathrm{i},\ \frac{3}{2}\big(1-\sqrt{3}\,\mathrm{i}\big),\ -\frac{3}{2}\big(1+\sqrt{3}\,\mathrm{i}\big)\right\}$

Question 2 : Let (a_n) be the sequence of real numbers defined recursively by

$$a_0 = 1 \qquad \text{and} \qquad a_{n+1} = 2a_n + 1 \qquad \text{for all } n \in \mathbb{N}.$$

Then

- ☐ $a_n = 2^{n+1} - 1\,,$ for all $n \in \mathbb{N}$
- ☐ $a_n = 1 + \frac{1}{3}n(n^2+5)\,,$ for all $n \in \mathbb{N}$
- ☐ $a_n = 2n + 1\,,$ for all $n \in \mathbb{N}$
- ☐ $a_n = (n+1)^2 - n\,,$ for all $n \in \mathbb{N}$

Question 3 : Let (a_n) be the sequence of real numbers defined by

$$a_n = \frac{\sqrt[3n]{4} - 6\,\sqrt[3n]{2} + 9}{\sqrt[2n]{9} - 4\,\sqrt[2n]{3} + 4}\,, \qquad \text{for all } n \in \mathbb{N}\setminus\{0\}\,.$$

Then

- ☐ the sequence diverges
- ☐ the sequence converges and $\lim\limits_{n\to+\infty} a_n = \dfrac{9}{4}$
- ☐ the sequence converges and $\lim\limits_{n\to+\infty} a_n = 4$
- ☐ the sequence converges and $\lim\limits_{n\to+\infty} a_n = \dfrac{4}{9}$

Question 4 : Let (a_n) be the sequence of real numbers defined recursively by

$$a_0 = 1 \qquad \text{and} \qquad a_{n+1} = \frac{a_n}{3} + \frac{a_n^2}{3} \qquad \text{for all } n \in \mathbb{N}.$$

Then

- ☐ the sequence converges and $\lim\limits_{n\to+\infty} a_n = \dfrac{2}{3}$
- ☐ the sequence does not converge
- ☐ the sequence converges and $\lim\limits_{n\to+\infty} a_n = 2$
- ☐ the sequence converges and $\lim\limits_{n\to+\infty} a_n = 0$

Question 5 : The series $\sum_{n=1}^{+\infty} \left(1 - \frac{1}{n^2}\right)^{n^2}$

- ☐ converges to a real number s such that $s < 3$
- ☐ diverges, but the alternating series $\sum_{n=1}^{+\infty} (-1)^n \left(1 - \frac{1}{n^2}\right)^{n^2}$ converges
- ☐ diverges
- ☐ converges to a real number s such that $s \geq 3$

Question 6 : Consider the series

$$\sum_{n=5}^{+\infty} \big(\cos(t\pi)\big)^n .$$

with parameter $t \in \mathbb{R}$. Then

- ☐ the series converges for all $t \notin \mathbb{Z}$
- ☐ the series diverges $t \in \mathbb{R}$
- ☐ the series converges for all $t \in \mathbb{R}$
- ☐ the series converges for a finite number of values of t

Question 7 : Let $p \in \mathbb{R}$ be any number.

- ☐ The series $\sum_{n=0}^{+\infty} \frac{\mathrm{Log}(n+2)}{(n+1)^p}$ diverges for all $p > 0$
- ☐ The series $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)^p}$ converges absolutely for all $p > 0$
- ☐ The series $\sum_{n=0}^{+\infty} \frac{1}{(n+1)^p (n+2)^p}$ converges for all $p > 0$
- ☐ The series $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)^p}$ converges for all $p > 0$

Question 8 : Let A be the non-empty bounded set $A = \left\{x \in [0, 4\pi] : \cos(x) < \frac{1}{4}\right\}$ and let $b = \operatorname{Sup} A$. Then

- ☐ $\cos(b) = \frac{1}{4}$
- ☐ $\cos(b) < \frac{1}{4}$
- ☐ $\sin(b) = \frac{1}{4}$
- ☐ $b < 2\pi$

Second part: true/false questions

For each question, cross the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 9 : Let A and B be non-empty bounded subsets of $\mathbb{R}$ such that $A \subset B$ and $A \neq B$. Then $\operatorname{Sup} A < \operatorname{Sup} B$.

☐ TRUE ☐ FALSE

Question 10 : Let (x_n) and (y_n) be two sequences of real numbers such that:

(i) $x_n \leq y_n$ for all n even, and

(ii) $x_n \geq y_n$ for all n odd.

If the sequence (x_n) converges, then the sequence (y_n) also converges.

☐ TRUE ☐ FALSE

Question 11 : Let (a_n) be a convergent sequence of real numbers. Then, for all $\varepsilon > 0$, there is a $k \in \mathbb{N}$, such that for all $n \geq k$ we have $|a_{n+1} - a_n| \leq \varepsilon$.

☐ TRUE ☐ FALSE

Question 12 : Let (x_n) and (y_n) be two bounded sequences of real numbers such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to+\infty} \sup x_n \leq \lim_{n\to+\infty} \sup y_n$.

☐ TRUE ☐ FALSE

Question 13 : If the series $\sum_{n=0}^{+\infty} a_n$ converges and (b_n) is a bounded sequence of real numbers, then the series $\sum_{n=0}^{+\infty} a_n b_n$ converges.

☐ TRUE ☐ FALSE

Question 14 : If a function $f\colon\,]0, +\infty[\to \mathbb{R}$ satisfies $\lim_{x\to 0+} \frac{f(x)}{x} = 0$, then the series $\sum_{n=1}^{+\infty} f\left(\frac{1}{n^2}\right)$ converges.

☐ TRUE ☐ FALSE