Analysis I (English)
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Analysis 1 - Exercise Set 10

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Calculate the derivative f' of the function f and give the domain of f and f'.

   (a) $f(x) = \dfrac{5x+2}{3x^2-1}$

   (b) $f(x) = \tan(x)$

   (c) $f(x) = x\sin(x) + \frac{\cos(x)^2}{x^2+2}$

Solution:

(a) We use the quotient rule to obtain $f'(x) = \dfrac{5(3x^2-1) - 6x(5x+2)}{(3x^2-1)^2} = -\dfrac{15x^2+12x+5}{(3x^2-1)^2}$; $D(f) = D(f') = \mathbb{R} \setminus \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$

(b) We apply the formula for the derivative of a quotient on $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, to obtain

$$f'(x) = \frac{\cos(x)^2 - \sin(x) \cdot \big(-\sin(x)\big)}{\cos(x)^2} = \frac{1}{\cos(x)^2}$$

So $D(f) = D(f') = \mathbb{R} \setminus \{x \in \mathbb{R} : \cos(x) = 0\} = \mathbb{R} \setminus \left\{\frac{(2k+1)\pi}{2} \ : \ k \in \mathbb{Z}\right\}$.

(c) We use the rules for product sum and quotient of derivatives.

$(x\sin(x))' = x\cos(x) + \sin(x)$,
$(\cos(x)^2)' = (\cos(x)\cos(x))' = \cos(x)(-\sin(x)) - \sin(x)\cos(x) = -2\sin(x)\cos(x) = -\sin(2x)$,
$(x^2+2)' = 2x$,

$$\begin{aligned} f'(x) &= x\cos(x) + \sin(x) + \frac{(x^2+2)(-\sin(2x)) - 2x\cos(x)^2}{(x^2+2)^2} \\ &= x\cos(x) + \sin(x) - \frac{\sin(2x)}{x^2+2} - \frac{2x\cos(x)^2}{(x^2+2)^2} \end{aligned}$$

So $D(f) = D(f') = \mathbb{R}$.

2. Let I be some open interval and $f : I \to \mathbb{R}$ be a function that is continuous at $x_0 \in I$. Prove that if $f(x_0) > 0$ then $f(x) > 0$ on some open interval containing x_0.

Solution:

Let $0 < \varepsilon < f(x_0)$. By continuity of f, $\exists \delta' > 0$ s.t. $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \quad \forall x \in (x_0 - \delta', x_0 + \delta') \cap I$. Since I is an open interval $I = (a, b)$, we can consider $\delta = \min\{\delta', |x_0 - a|, |x_0 - b|\}$ to guarantee that $(x_0 - \delta, x_0 + \delta) \subset I$. Then, it follows that $f(x) > f(x_0) - \varepsilon > 0 \quad \forall x \in (x_0 - \delta, x_0 + \delta)$, as required.

3. Prove the quotient rule for derivatives:
if $f : I \to \mathbb{R}$, $f(x) = \frac{g(x)}{h(x)}$, $x_0 \in I$ and both g and h are differentiable at x_0, with $h(x_0) \neq 0$, then, $f'(x_0) = \frac{g'(x_0)h(x_0) - g(x_0)h'(x_0)}{h(x_0)^2}$.

Solution:

For all $x_0 \in I \setminus \{x : h(x) = 0\}$, we have

$$\begin{aligned} f'(x_0) &= \lim_{s \to 0} \frac{f(x_0 + s) - f(x_0)}{s} \\ &= \lim_{s \to 0} \frac{\frac{g(x_0+s)}{h(x_0+s)} - \frac{g(x_0)}{h(x_0)}}{s} \\ &= \lim_{s \to 0} \frac{g(x_0 + s)h(x_0) - g(x_0)h(x_0 + s)}{sh(x_0)h(x_0 + s)}. \end{aligned}$$

Here we are relying on the continuity of h: indeed, if $h(x_0) \neq 0$, by continuity, $h(x_0+s) \neq 0$ for s small (cf. Exercise 2 in this worksheet).

Since g, h are differentiable at x_0, then they are also continuous at that point, hence $\lim\limits_{s \to 0} \frac{1}{h(x)h(x+s)} = \frac{1}{h(x)^2}$. Moreover,

$$\begin{aligned} &\lim_{s \to 0} \frac{g(x_0 + s)h(x_0) - g(x_0)h(x_0 + s)}{s} \\ &= \lim_{s \to 0} \frac{g(x_0 + s)h(x_0) - g(x_0)h(x_0) + g(x_0)h(x_0) - g(x_0)h(x_0 + s)}{s} \\ &= \lim_{s \to 0} \left(h(x_0) \frac{g(x_0 + s) - g(x_0)}{s} - g(x_0) \frac{h(x_0 + s) - h(x_0)}{s} \right) \\ &= (g'(x_0)h(x_0) - g(x_0)h(x_0)). \end{aligned}$$

Hence,

$$f'(x_0) = \frac{g'(x_0)h(x_0) - g(x_0)h(x_0)}{h(x_0)^2}.$$

4. For each of the following functions, find the inverse function and the derivative of the inverse function.

(a) $f(x) = \cos x$, $x \in]0, \pi[$.

(b) $f(x) = \tan x$, $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Solution:

(a) The inverse function is given by $g(x) = f^{-1}(x) = \arccos(x)$. Since $f'(x) = -\sin x$,

$$f'(g(x)) = -\sin(\arccos(x))$$

To find $\sin(\arccos(x))$, let $\theta = \arccos(x)$. We want to find $\sin\theta$. We have

$$\theta = \arccos(x) \Rightarrow \cos\theta = x \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2},$$

where in the last step we used that $\theta \in]0,\pi[$, which guarantees that $\sin\theta > 0$. Putting everything together

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{1}{\sqrt{1-x^2}}.$$

(b) The inverse function is given by $g(x) = f^{-1}(x) = \arctan(x)$. Since $f'(x) = \frac{1}{\cos^2 x}$,

$$f'(g(x)) = \cos^2(\arctan(x))$$

To find $\cos^2(\arctan(x))$, let $\theta = \arctan(x)$. We want to find $\cos\theta$. We have

$$\begin{aligned}\theta = \arctan(x) \Rightarrow \tan\theta = x &\Rightarrow \frac{\sin\theta}{\cos\theta} = x \\ &\Rightarrow \frac{\sin^2\theta}{\cos^2\theta} = x^2 \\ &\Rightarrow \frac{\sin^2\theta + \cos^2\theta - \cos^2\theta}{\cos^2\theta} = x^2 \\ &\Rightarrow \frac{1}{\cos^2\theta} - 1 = x^2 \\ &\Rightarrow \cos^2\theta = \frac{1}{1+x^2}\end{aligned}$$

Putting everything together

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+x^2}.$$

5. Calculate the derivative f' of the function f and give the domain of f and f'.

(a) $f(x) = \dfrac{x^2}{\sqrt{1-x^2}}$

(b) $f(x) = \sin(x)^2 \cdot \cos(x^2)$

(c) $f(x) = \sqrt{\sin\left(\sqrt{\sin(x)}\right)}$

(d) $f(x) = \sin(x)\log(\sin(x))e^{\cos(x)}$

Solution:

(a) We use the derivative rules of quotient and of composition of functions

$$f'(x) = \frac{2x\sqrt{1-x^2} - x^2\frac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2} = \frac{x(2-x^2)}{(1-x^2)^{3/2}} \;; \quad D(f) = D(f') =]-1,1[.$$

(b) We use the derivative rules of multiplication and of composition of functions

$$\begin{aligned} f'(x) &= 2\sin(x)\cos(x)\cdot\cos(x^2) + \sin(x)^2\cdot\big(-\sin(x^2)\big)\cdot 2x \\ &= 2\sin(x)\big(\cos(x)\cos(x^2) - x\sin(x)\sin(x^2)\big)\,; \qquad D(f) = D(f') = \mathbb{R}. \end{aligned}$$

(c) We use the derivative rule of composition of functions

$$\begin{aligned} f'(x) &= \frac{1}{2\cdot\sqrt{\sin\Big(\sqrt{\sin(x)}\Big)}}\cos\Big(\sqrt{\sin(x)}\Big)\,\frac{1}{2\cdot\sqrt{\sin(x)}}\cos(x) \\ &= \frac{\cos\Big(\sqrt{\sin(x)}\Big)\cos(x)}{4\cdot\sqrt{\sin\Big(\sqrt{\sin(x)}\Big)}\cdot\sqrt{\sin(x)}}\,. \end{aligned}$$

The domain of f is

$$D(f) = \Big\{x\in\mathbb{R} : \sin(x)\geq 0 \text{ and } \sin\Big(\sqrt{\sin(x)}\Big)\geq 0\Big\} = \bigcup_{k\in\mathbb{Z}}\big[2k\pi,(2k+1)\pi\big]\,.$$

in fact, $\sin(x)\geq 0 \;\Leftrightarrow\; x\in\big[2k\pi,(2k+1)\pi\big]$ and for these values, we have $\sqrt{\sin(x)}\in[0,1]$ so $\sin\Big(\sqrt{\sin(x)}\Big)\geq 0$, which means f is well defined.

For the domain of f', we need to exclude the points where $\sin(x)=0$, so

$$D(f') = \bigcup_{k\in\mathbb{Z}}\big]2k\pi,(2k+1)\pi\big[\,.$$

(d)

$$\begin{aligned} f'(x) &= \cos(x)\log(\sin(x))e^{\cos(x)} + \sin(x)\frac{1}{\sin(x)}\cos(x)e^{\cos(x)} \\ &\quad + \sin(x)\log(\sin(x))e^{\cos(x)}(-\sin(x)) \\ &= e^{\cos(x)}(\log(\sin(x))(\cos(x)-\sin^2(x)) + \cos(x)). \end{aligned}$$

So $D(f) = D(f') = \bigcup_{k\in\mathbb{Z}}]2k\pi,(2k+1)\pi[$, because $\log(x)$ is defined only for $x>0$.

6. For $x\in\mathbb{R}$, e^x has been defined as $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$. Hence, this definition gives rise to a function $f\colon\mathbb{R}\to\mathbb{R}$, $f(x)=e^x$. Prove the following properties of the exponential e^x:

(a) $e^0 = 1$;

(b) $e^x\cdot e^y = e^{x+y}$; [For this part of the exercise you can assume the following result: *Let $(a_n),(b_n)$ be sequences. Assume that both $\sum\limits_{i=0}^{\infty}a_i, \sum\limits_{i=0}^{\infty}b_i$ converge to a finite limit, and, moreover, that at least one of $\sum\limits_{i=0}^{\infty}a_i, \sum\limits_{i=0}^{\infty}b_i$ converges absolutely. Then the sequence (z_n), $z_n := \sum_{l=0}^{n}a_l b_{n-l}$ satisfies*

$$\sum_{i=0}^{\infty}a_i\cdot\sum_{i=0}^{\infty}b_i = \sum_{i=0}^{\infty}z_i.$$

(c) $e^{-x} = \frac{1}{e^x}$

(d) e^x is a strictly increasing function of x; e^{-x} is a strictly decreasing function of x;

(e) Use the definition of $\log(x)$ as inverse of the function e^x to show that

  (i) $\log(ab) = \log(a) + \log(b)$ for all $a, b > 0$.

  (ii) $\log(a^b) = b\log(a)$ for all $a > 0$ and all $b \in \mathbb{R}$.

  (iii) $\log(x)$ is a strictly increasing function of x.

Solution:

(a) $f(0) = e^0 = 1 + \frac{0}{1!} + \frac{0^2}{2!} + \cdots = 1$

(b) By D'Alembert's criterion, for any $x \in \mathbb{R}$, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent, since

$$\lim_{n\to\infty} \frac{\frac{|x^n|}{n!}}{\frac{|x^{n-1}|}{(n-1)!}} = \lim_{n\to\infty} \frac{|x|}{n} = 0.$$

Note that $\forall x, y \in \mathbb{R}$,

$$\frac{(x+y)^n}{n!} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k y^{n-k}}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}.$$

Hence, $\forall x, y \in \mathbb{R}$,

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}),$$

but the latter term is exactly equal to

$$\sum_{n=0}^{\infty} z_n, \quad \text{where } z_n := \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}.$$

The result that was cited in the text of the exercise and we are free to assume then implies that

$$\sum_{n=0}^{\infty} z_n = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right),$$

that is, $e^{x+y} = e^x \cdot e^y$.

(c) By (a), we have $e^0 = 1$. Now, fix $x \in \mathbb{R}$. By (b), we have

$$e^x \cdot e^{-x} = e^{x-x} = e^0 = 1.$$

So, this proves that $e^x \neq 0$, $e^{-x} \neq 0$, and that $e^{-x} = \frac{1}{e^x}$. As a byproduct, it follows that $e^x > 0$ for all x: indeed, this is clear from the definition if $x > 0$, as each summand $\frac{x^n}{n!} > 0$. Then, the case $x = 0$ follows from (a), while the case $x < 0$ follows from the case $x > 0$ and part (c) we just proved.

(d) If $a, b \in \mathbb{R}$ such that $a < b$, then $e^a < e^b$ holds if and only if $e^{b-a} > 1$ by using (c). But

$$e^{b-a} = \sum_{k=0}^{\infty} \frac{(b-a)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(b-a)^k}{k!} > 1$$

because $b - a > 0$. Now to show that e^{-x} is a decreasing function it suffices to note that e^{-x} is a the reciprocal of an increasing function.

(e) (i) $e^{\log(ab)} = ab = e^{\log(a)}e^{\log(b)} = e^{\log(a)+\log(b)}$. Since e^x is strictly increasing, it is injective; thus, we conclude that $\log(ab) = \log(a) + \log(b)$.

(ii) $e^{\log(a^b)} = a^b = (e^{\log(a)})^b = e^{b\log(a)}$. Since e^x is injective, we conclude that $\log(a^b) = b\log(a)$.

(iii) $\log(x)$ is the inverse function of a strictly increasing function, hence, it is strictly increasing.

7. For each function, calculate $f^{(n)}$, the n-th order derivative of f.

(a) $f(x) = x^m \quad (m \in \mathbb{Z})$

(b) $f(x) = \sin(2x) + 2\cos(x)$

(c) $f(x) = \log(x)$

Solution:

(a) We identify three cases for m:

- $m = 0$: $\quad f^{(n)}(x) = 0$ for all $n \in \mathbb{N}^*$.
- $m \geq 1$: $\quad f^{(n)}(x) = \begin{cases} m(m-1)(m-2)\cdots(m-n+1)x^{m-n}, & n \leq m \\ 0, & n > m \end{cases}$ The first case follows immediately, while the second and the third can be proved by induction on m (resp. $-m$).
- $m \leq -1$: $\quad f^{(n)}(x) = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$ for all $n \in \mathbb{N}^*$

(b) We start by calculating the first four derivatives of f:

$$f'(x) = 2\cos(2x) - 2\sin(x) \qquad f''(x) = -4\sin(2x) - 2\cos(x)$$
$$f'''(x) = -8\cos(2x) + 2\sin(x) \qquad f^{(4)}(x) = 16\sin(2x) + 2\cos(x)$$

We need to distinguish two cases for $n \in \mathbb{N}^*$:

$$f^{(n)}(x) = \begin{cases} (-1)^{\frac{n}{2}}\left(2^n \sin(2x) + 2\cos(x)\right), & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\left(2^n \cos(2x) - 2\sin(x)\right), & n \text{ odd} \end{cases}$$

(c) Since $f'(x) = x^{-1}$, we can use the result in part (a) with $m = -1$ to obtain $f^{(n)}$. We have,

$$f^{(n)}(x) = (f')^{(n-1)}(x) = (-1)(-2)(-3)\cdots(-(n-1))x^{-1-(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. State if the following are true or false.

(a) f even $\quad\Rightarrow\quad f'$ odd,

(b) f odd $\quad\Rightarrow\quad f'$ even,

(c) f' even $\quad\Rightarrow\quad f$ odd,

(d) f periodic $\quad\Rightarrow\quad f'$ periodic.

Solution:

(a) True. We have $f(-x) = f(x)$. Using the derivative of composite functions we obtain $-f'(-x) = f'(x)$, so f' is odd.

(b) True. We take the derivative of $f(-x) = -f(x)$ to obtain $-f'(-x) = -f'(x) \Leftrightarrow f'(-x) = f'(x)$. So f' is even.

(c) False. Take for example $f(x) = x + 1$.

(d) True. For a periodic function f, there exists $T > 0$ such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. By taking the derivative, we have $f'(x + T) = f'(x)$ and so f' is also periodic.

9. Find maximum and minimum of the following functions

(a) $f(x) = x$ in $[-\pi, \pi]$

(b) $f(x) = \sin(x) + \cos(x)$ in $[0, \frac{2\pi}{3}]$

Solution:

(a) The function is strictly increasing, so the minimum is at $-\pi$ and the maximum at π

(b) The derivative is $f'(x) = \cos(x) - \sin(x)$ and it vanishes only in $x = \frac{\pi}{4}$. We compute

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad f(0) = 1, \quad f\left(\frac{2\pi}{3}\right) = \frac{-1 + \sqrt{3}}{2}$$

So there is a maximum in $\frac{\pi}{4}$, and a minimum in $\frac{2}{3}\pi$.

10. Calculate $(g \circ f)'(0)$ for the functions $f, g \colon \mathbb{R} \to \mathbb{R}$ defined by

(a) $f(x) = 2x + 3 + (e^x - 1)\sin(x)^7\cos(x)^4$ and $g(x) = \log(x)^3$.

(b) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 2x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = (x - 1)^4$.

Solution: Since $(g \circ f)'(0) = g'(f(0)) \cdot f'(0)$, we need to find the derivatives of f and g.

(a) To find $f'(x)$, we write $f(x) = 2x + 3 + (e^x - 1)\,u(x)$ where $u(x) = \sin(x)^7\cos(x)^4$. Then

$$f'(x) = 2 + e^x\,u(x) + (e^x - 1)\,u'(x) \quad \text{and} \quad u'(x) = 7\sin(x)^6\cos(x)^5 - 4\sin(x)^8\cos(x)^3.$$

We have $u(0) = u'(0) = 0$ and so $f'(0) = 2$.

Then we have $g'(x) = \dfrac{3\log(x)^2}{x}$. Since $f(0) = 3$ we finally show that

$$(g \circ f)'(0) = g'(3) \cdot f'(0) = \frac{3\log(3)^2}{3} \cdot 2 = 2\log(3)^2.$$

(b) For calculating $f'(0)$, we must use the definition of derivative:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) + 2x - 0}{x} = \lim_{x\to 0} \left(x \sin\left(\tfrac{1}{x}\right) + 2\right) = 2$$

This is because $\lim_{x\to 0} \left(x \sin\left(\frac{1}{x}\right)\right) = 0$, as the function sin is bounded, and so $x \sin\left(\frac{1}{x}\right)$ is squeezed to 0 by the factor x as $x \to 0$.

Since $g'(x) = 4(x-1)^3$ and $f(0) = 0$, we obtain

$$(g \circ f)'(0) = g'(0) \cdot f'(0) = (-4) \cdot 2 = -8.$$

11. Calculate the derivative f' of the function f and give the domain of f and f'.

(a) $f(x) = \sqrt[5]{\left(2x^4 + e^{-(4x+3)}\right)^3}$

(b) $f(x) = e^{\sqrt[3]{\log(4x)^2}}$

(c) $f(x) = \log\left(4^{\sin(x)}\right) e^{\cos(4x)}$

Solution:

(a) $f'(x) = \frac{3}{5}\left(2x^4 + e^{-(4x+3)}\right)^{-2/5}\left(8x^3 - 4e^{-(4x+3)}\right) = \dfrac{12\left(2x^3 - e^{-(4x+3)}\right)}{5\sqrt[5]{\left(2x^4 + e^{-(4x+3)}\right)^2}}$;

$D(f) = D(f') = \mathbb{R}$ (The denominator of f' is nonzero since $e^{-(4x+3)} > 0$ and $x^4 \geq 0$ for all $x \in \mathbb{R}$.)

(b)

$$f'(x) = e^{\sqrt[3]{\log^2(4x)}} \frac{2}{3} (\log(4x))^{-\frac{1}{3}} \frac{1}{4x} \cdot 4 = \frac{2}{3} \frac{e^{\sqrt[3]{\log^2(4x)}}}{x\sqrt[3]{\log(4x)}}$$

The domain of f is $]0, +\infty[$ because $\log(x)$ is defined only for $x > 0$. For the domain of f' we have to exclude all the points where the denominator vanishes, that is $x = \frac{1}{4}$, because $\log(4x) = 0$ if and only if $4x = 1$. So $D(f') =]0, \frac{1}{4}[\cup]\frac{1}{4}, +\infty[$.

(c) We have $f(x) = \sin(x) \log(4)\, e^{\cos(4x)}$ by Exercise 6. We obtain

$$\begin{aligned} f'(x) &= \log(4) \cos(x)\, e^{\cos(4x)} + \log(4) \sin(x) \cdot \big(-4 \sin(4x)\big) \cdot e^{\cos(4x)} \\ &= \log(4)\, e^{\cos(4x)} \Big(\cos(x) - 4 \sin(x) \sin(4x) \Big). \end{aligned}$$

$D(f) = D(f') = \mathbb{R}$

12. State if the following are true or false.

(a) If f is differentiable at $a \in \mathbb{R}$, Then there is $\delta > 0$ such that f is continuous on $]a-\delta, a+\delta[$.

(b) If f is differentiable from left and right at $a \in \mathbb{R}$, then f is differentiable at a.

(c) If f is differentiable on $\mathbb{R}$, then $g(x) = \sqrt{f^2(x)}$ is differentiable on $\mathbb{R}$.

Solution:

(a) False. Take for example $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0\,, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. This function is continuous at $x = 0$ because,

$$0 \leq f(x) \leq x^2$$

for all $x \in \mathbb{R}$. So by the squeeze theorem we have $\lim_{x \to 0} f(x) = 0 = f(0)$. On the other hand f is not continuous at any point other than 0. In fact, let $x_0 \in \mathbb{R}$, $x_0 \neq 0$. For $n \in \mathbb{N}^*$, The open interval $\left]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right[$ contains rational and irrational numbers. We take $a_n \in \left]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right[\cap \mathbb{Q}$ and $b_n \in \left]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right[\cap (\mathbb{R} \setminus \mathbb{Q})$ for each $n \in \mathbb{N}^*$. Then the two sequences $(a_n) \subset \mathbb{Q}$ and $(b_n) \subset \mathbb{R} \setminus \mathbb{Q}$ both converge to x_0, but

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n^2 = x_0^2 > 0 = \lim_{n \to \infty} f(b_n),$$

so f is not continuous at x_0.

But f is differentiable at $x = 0$. Indeed, we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and since $-|x| \leq \frac{f(x)}{x} \leq |x|$ for all $x \in \mathbb{R}$, the squeeze theorem gives

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0\,.$$

(b) False. Take for example $f(x) = |x|$ which is not differentiable at 0. But the left and right derivatives exist (Look at the lecture notes).

(c) False. By taking $f(x) = x$, we have $g(x) = \sqrt{x^2} = |x|$ which is not differentiable at 0.

13. For each of the following functions, find the inverse function. Find the derivative of the inverse function once by direct calculation and once by the inverse function derivative.

(a) $f(x) = \sqrt{x^2 + 9}$, $x \geq 0$.

(b) $f(x) = \frac{1}{1+x}$, $x \neq -1$.

Solution:

(a) To find $g(x) = f^{-1}$, solve $y = \sqrt{x^2 + 9}$ for x. This yields $x = \pm\sqrt{y^2 - 9}$. Because the domain of f is restricted to $x \geq 0$, we must choose the positive sign in front of the radical. Thus

$$g(x) = f^{-1}(x) = \sqrt{x^2 - 9}$$

By the formula for the derivative of the inverse function we have

$$g'(x) = \frac{1}{f'(g(x))}$$

with

$$f'(x) = \frac{x}{\sqrt{x^2 + 9}}$$

so

$$f'(g(x)) = \frac{\sqrt{x^2-9}}{\sqrt{(\sqrt{x^2-9})^2+9}} = \frac{\sqrt{x^2-9}}{\sqrt{x^2}} = \frac{\sqrt{x^2-9}}{x}$$

since the domain of g is $x \geq 3$. Thus,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{x}{\sqrt{x^2-9}}$$

This agrees with the answer we obtain by differentiating directly:

$$g'(x) = \frac{2x}{2\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}$$

(b) To find $g(x) = f^{-1}$, solve $y = \frac{1}{1+x}$ for x. This yields $x = \frac{1-y}{y}$. Thus

$$g(x) = \frac{1-x}{x}$$

By direct calculation we can rewrite $g(x) = x^{-1} - 1$. So we see that $g'(x) = -x^{-2}$. Also we see that $f'(x) = -(1+x)^{-2}$, so

$$f'(g(x)) = -\left(1 + \frac{1-x}{x}\right)^{-2} = -(x^{-1})^{-2} = -x^2$$

and

$$g'(x) = \frac{1}{f'(g(x))} = -x^{-2}.$$

14. Show that the derivative of the function

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at $x = 0$ is zero and then find $f'(x)$. Is f' continuous?

Solution: We use the definition of derivative, So we write

$$x^2 \cos \frac{1}{x} = 0 + 0 \cdot (x - 0) + r(x)$$

And we must show that $\lim_{x\to 0} \frac{r(x)}{x-0} = 0$

$$\lim_{x\to 0} \frac{r(x)}{x} = \lim_{x\to 0} \frac{x^2 \cos \frac{1}{x}}{x} = \lim_{x\to 0} x \cos \frac{1}{x} = 0$$

And the final step holds because cos is a bounded function. Now since $f(x)$ has no singularities on $]-\infty, 0[$ and $]0, \infty[$ we may use the derivative formulas to compute $f'(x)$.

$$f'(x) = \begin{cases} 2x \cos \frac{1}{x} + \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We see that $\lim_{x\to 0^-} f'(x)$ and $\lim_{x\to 0^+} f'(x)$ do not exist (why?) even though $f'(0)$ does exist. It means that f is a differentiable function but its derivative is not continuous at $x = 0$.

15. Find maximum and minimum of the following functions

(a) $f(x) = x^2 - 5$ in $[-\pi, \pi]$

(b) $f(x) = \sqrt[3]{(x-1)(x-2)^2}$ in $[1 + \frac{1}{10}, 2 - \frac{1}{10}]$

Solution:

(a) The derivative is $2x$, and vanishes at 0. We compute

$$f(0) = -5, \quad f(-\pi) = f(\pi) > 0.$$

So the minimum is at 0 and there are two maxima, at $-\pi$ and π.

(b) The derivative is

$$f'(x) = \frac{(x-2)(3x-4)}{(3(x-1)(x-2)^2)^{\frac{2}{3}}}$$

This is zero at $x_0 = 4/3$. We compute

$$f\left(\frac{4}{3}\right) = \frac{\sqrt[3]{4}}{3}, \quad f\left(1 + \frac{1}{10}\right) = \frac{3\sqrt[3]{3}}{10}, \quad f\left(2 - \frac{1}{10}\right) = \frac{\sqrt[3]{9}}{10}.$$

So we have a maximum at $\frac{4}{3}$ and a minimum at $2 - \frac{1}{10}$.

16. Calculate f'

(a) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{\cos x}{2+\sin(\log x)}$

(b) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \log(a|x|)$, $a > 0$

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x^2 \sin x}$

Solution:

(a)

$$f'(x) = \frac{-\sin(x)\ (2 + \sin(\log x)) - \frac{1}{x}\cos x \cos(\log x)}{(2 + \sin(\log x))^2}$$

(b) If $x > 0$ we have $f(x) = \log(ax)$ and so,

$$f'(x) = \frac{a}{ax} = \frac{1}{x}$$

If $x < 0$ we have $f(x) = \log(-ax)$ and so

$$f'(x) = \frac{-a}{-ax} = \frac{1}{x}$$

So, we can say that that $f'(x) = \frac{1}{x}$ for all $x \in \mathbb{R} \setminus \{0\}$

(c) $f'(x) = (2x \sin x + x^2 \cos x)e^{x^2 \sin x}$

17. State if the following are true or false.

(a) If $f : E \to F$ is strictly increasing and bijective, then the inverse function is strictly increasing.
(b) If $f(x) = x^2 - 2x$, then $(f \circ f)'(1) = 0$.
(c) If a car traveled 210 km in 3 hours, then the speedometer must have read 70 km/h at least once.

Solution:

(a) True. If $a < b$ in F, then $f^{-1}(a) \neq f^{-1}(b)$ because f^{-1} is injective. Moreover, if $f^{-1}(a) > f^{-1}(b)$, then $a = f(f^{-1}(a)) > f(f^{-1}(b)) = b$ which is a contradiction.

(b) True. We have $f'(1) = 2 - 2 = 0$ and then $(f \circ f)'(1) = f'(f(1)) \cdot f'(1) = 0$.

(c) True. Let $f(t)$ be the traveled distance (Km) of the car at time t (h). Then $f'(t)$ is the speed of the car at time t. Now we apply the mean value theorem on the interval $[0, 3]$h. There should a time T such that

$$f'(T) = \frac{f(3) - f(0)}{3} = \frac{210}{3} = 70(\text{Km/h})$$

18. Find the inverse of the following functions if they exist. Give the domain of both functions.

(a) $f(x) = \left(\frac{1}{8}\right)^{1-x}$
(b) $f(x) = \log x - \log 2x + \log 3x$

Solution:

(a) The domain of f is all the real numbers $D_f = \mathbb{R}$. To find the inverse function we have:

$$y = \left(\frac{1}{8}\right)^{1-x} \Rightarrow \log y = (1-x)\log(\frac{1}{8}) \Rightarrow x = 1 - \frac{\log y}{\log(\frac{1}{8})}$$

So the inverse function is given by $f^{-1}(x) = 1 - \frac{\log x}{\log(\frac{1}{8})}$. Noting that the argument of the logarithm should be strictly positive the domain of the inverse function is $D_{f^{-1}} =]0, \infty[$.

(b) The domain of f is $D_f =]0, \infty[$. To find the inverse function first note that $f(x) = \log \frac{3x}{2}$. So we have

$$y = \log \frac{3x}{2} \Rightarrow e^y = \frac{3x}{2} \Rightarrow x = \frac{2}{3}e^y$$

So the inverse function is given by $f^{-1}(x) = \frac{2}{3}e^x$ and $D_{f^{-1}} = \mathbb{R}$.

19. Compute

$$\lim_{x \to +\infty} \log(x).$$

Solution: $\lim_{x \to +\infty} \log(x)$ is the supremum of the range of the function $\log(x)$, that is, the supremum of the domain of its inverse, which is e^x. So

$$\lim_{x \to +\infty} \log(x) = \lim_{x \to +\infty} \log(x) = \sup \mathbb{R} = +\infty.$$

20. The limit

$$\lim_{n\to\infty} \frac{\cos(n)}{\log(n)}$$

is

(a) 0

(b) -1

(c) $+1$

(d) $+\infty$

Solution: (a) is correct. Note that

$$\frac{-1}{\log(n)} \leq \frac{\cos(n)}{\log(n)} \leq \frac{1}{\log(n)}$$

And $\log(n) \to \infty$ as $n \to \infty$. So by squeeze theorem, the limit of the sequence is 0.