Lecturer: T. Mountford
Analysis I - (n/a)
15th January 2024
3h30

n/a

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SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 16 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  $+3$ points if your answer is correct,
  0 points if you give no answer or more than one,
  -1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  $+1$ points if your answer is correct,
  0 points if you give no answer or more than one,
  -1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $(a_n)_{n\geq 1}$ be the sequence defined by

$$a_n = (-1)^{n+1} + \left(-\frac{1}{2}\right)^n + \frac{3}{n}.$$

Then :

- ☐ $\liminf_{n\to\infty} a_n = -1$ and $\limsup_{n\to\infty} a_n = 1$
- ☐ $\liminf_{n\to\infty} a_n = \frac{3}{4}$ and $\limsup_{n\to\infty} a_n = \frac{7}{2}$
- ☐ $\liminf_{n\to\infty} a_n = -\frac{1}{4}$ and $\limsup_{n\to\infty} a_n = \frac{3}{2}$
- ☐ $\liminf_{n\to\infty} a_n = -1$ and $\limsup_{n\to\infty} a_n = \frac{3}{2}$

Question 2 : Let $(a_n)_{n\geq 1}$ be the sequence defined by $a_n = (-1)^n + \frac{1}{n}$, and let $A = \{a_1, a_2, a_3, \dots\}$. Then :

- ☐ $\inf A = 0$ and $\sup A = 1 + \frac{1}{2}$
- ☐ $\inf A = -1$ and $\sup A = 1$
- ☐ $\inf A = -1$ and $\sup A = 1 + \frac{1}{2}$
- ☐ $\inf A = 0$ and $\sup A = 1$

Question 3 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$$

Then :

- ☐ $f'(0) = 1$
- ☐ $f'(0) = \frac{1}{2}$
- ☐ f is not differentiable at 0
- ☐ $f'(0) = e$

Question 4 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2^x + x^2$. Then :

- ☐ there exists $c \in]2, 3[$ such that $f'(c) = 9$.
- ☐ there exists $c \in]0, 1[$ such that $f'(c) = 9$.
- ☐ there exists $c \in]3, 4[$ such that $f'(c) = 9$.
- ☐ there exists $c \in]1, 2[$ such that $f'(c) = 9$.

Question 5 : The integral $\displaystyle\int_0^{\pi} e^x \cos(2x)\,\mathrm{d}x$ equals

- ☐ $\frac{1}{5}(e^{\pi} - 1)$
- ☐ $\frac{2}{5}(e^{\pi} - 1)$
- ☐ $e^{\pi} - 1$
- ☐ 0

Question 6 : Let I be a nonempty interval of $\mathbb{R}$, and $f\colon I \to \mathbb{R}$ a function and $\mathrm{Im}(f)$ is the range of f. Among the statements below, which is always true?

- ☐ If I is closed, bounded and if $\mathrm{Im}(f)$ is open, then f is not continuous on I.
- ☐ If I is bounded and if $\mathrm{Im}(f)$ is closed and if f is continuous on I, then I is closed.
- ☐ If I is closed and bounded and if $\mathrm{Im}(f)$ is closed, then, f is continuous on I.
- ☐ If I is bounded and if $\mathrm{Im}(f)$ is bounded, then f is continuous on I.

Question 7 : Consider the series with parameter $x \in]0,1[\cup]1,+\infty[$ defined by

$$\sum_{n=1}^{\infty} \frac{1}{(\log(x))^n}.$$

Then, the series converges if and only if

- ☐ $x \in \left]0, \frac{1}{e}\right[$
- ☐ $x \in \left]0, \frac{1}{e}\right[\cup]e,+\infty[$
- ☐ $x \in]e,+\infty[$
- ☐ $x \in \left]\frac{1}{e}, 1\right[\cup]1,e[$

Question 8 : The integral $\displaystyle\int_0^2 \frac{1}{x^2+3x+2}\,\mathrm{d}x$ equals

- ☐ $\log(6)$
- ☐ $\log\left(\frac{3}{8}\right)$
- ☐ $\log\left(\frac{4}{3}\right)$
- ☐ $\log\left(\frac{3}{2}\right)$

Question 9 : One of the solutions of the equation $z^5 = \left(1+\sqrt{3}\,\mathrm{i}\right)^2$ is

- ☐ $z = \sqrt[5]{4}\,\left(\cos\left(\frac{2\pi}{15}\right) + \mathrm{i}\sin\left(\frac{2\pi}{15}\right)\right)$
- ☐ $z = \sqrt[5]{4}\,\left(\cos\left(\frac{16\pi}{15}\right) + \mathrm{i}\sin\left(\frac{16\pi}{15}\right)\right)$
- ☐ $z = \sqrt[5]{2}\,\left(\cos\left(\frac{16\pi}{15}\right) + \mathrm{i}\sin\left(\frac{16\pi}{15}\right)\right)$
- ☐ $z = \sqrt[5]{2}\,\left(\cos\left(\frac{2\pi}{15}\right) + \mathrm{i}\sin\left(\frac{2\pi}{15}\right)\right)$

Question 10 : Let $f\colon [0,\pi] \to \mathbb{R}$ be the function defined by

$$f(x) = (x+1)\sin(x) + \cos(x) + e^{\sin(x)}.$$

Then, the range of f is

- ☐ $[0, 2+\pi+e]$
- ☐ $\left[0, 1+\frac{\pi}{2}+e\right]$
- ☐ $[0,2]$
- ☐ $[\pi-2, 2]$

Question 11 : The interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{4^n}{n+1}(x-1)^n$$

is

- ☐ $\left]\frac{3}{4}, \frac{5}{4}\right[$
- ☐ $\left]\frac{1}{2}, \frac{3}{2}\right[$
- ☐ $\left[\frac{3}{4}, \frac{5}{4}\right[$
- ☐ $\left[\frac{1}{2}, \frac{3}{2}\right[$

Question 12 : Let $(x_n)_{n\geq 1}$ be the sequence defined by

$$x_n = \left(\cos\left(\sqrt{\tfrac{2}{n}}\right)\right)^n.$$

Then the limit $\lim_{n\to\infty} x_n$ equals

- ☐ 1
- ☐ $\frac{1}{e}$
- ☐ e
- ☐ 0

Question 13 :
Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{|x|} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$$

Then :

- ☐ f is continuous on $\mathbb{R}$, but not differentiable at $x = 0$
- ☐ f is differentiable at $x = 0$
- ☐ f is differentiable from the right at $x = 0$
- ☐ $\lim\limits_{x\to 0} f(x)$ exists but f is not continuous at $x = 0$

Question 14 : Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = e^{1+x-\cos(x)}$. The expansion to order 3 for f around $x_0 = 0$ is given by

- ☐ $f(x) = 1 - x + x^2 - \frac{2}{3}x^3 + x^3\varepsilon_3(x)$
- ☐ $f(x) = 1 + x + x^2 + \frac{2}{3}x^3 + x^3\varepsilon_3(x)$
- ☐ $f(x) = 1 - x + \frac{1}{3}x^3 + x^3\varepsilon_3(x)$
- ☐ $f(x) = 1 + x - \frac{1}{3}x^3 + x^3\varepsilon_3(x)$

Question 15 : Let $(u_n)_{n\geq 0}$ be the sequence defined by $u_0 = 1$ and, for $n \geq 1$, $u_n = -\frac{2}{3}u_{n-1} + 2$. Then :

- ☐ $\lim\limits_{n\to\infty} u_n = +\infty$
- ☐ $\lim\limits_{n\to\infty} u_n = \frac{6}{5}$
- ☐ $\lim\limits_{n\to\infty} u_n = -\infty$
- ☐ $\lim\limits_{n\to\infty} u_n = 2$

Question 16 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \left|4 - x^2\right| & \text{if } x \leq 0\,, \\ 4\left|x^2 - 1\right| & \text{if } x > 0\,. \end{cases}$$

Then :

- ☐ f is not continuous at $x = 1$
- ☐ f is not continuous at $x = 0$
- ☐ f is not continuous at $x = -2$
- ☐ f is continuous on $\mathbb{R}$

Question 17 : Let, for $k \in \mathbb{N}^*$, $a_k = (-1)^k \dfrac{k+2}{k^3}$ and let $s_n = \displaystyle\sum_{k=1}^{n} a_k$. Then :

- ☐ the series $\displaystyle\sum_{k=1}^{+\infty} a_k$ converges absolutely.
- ☐ $\displaystyle\lim_{n\to\infty} s_n = -\infty$.
- ☐ $\displaystyle\lim_{n\to\infty} s_n = +\infty$
- ☐ the series $\displaystyle\sum_{k=1}^{+\infty} a_k$ converges, but not absolutely.

Question 18 : The generalized integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$ equals

- ☐ $\dfrac{\pi}{2}$
- ☐ $2\arctan(e)$
- ☐ 1
- ☐ $\arctan\left(\dfrac{1}{2}\right)$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x \to +\infty} f(x) = +\infty$ and let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = 1$ and, for $n \geq 1$, $a_n = f(a_{n-1})$. Then $\lim_{n\to\infty} a_n = +\infty$.

☐ TRUE ☐ FALSE

Question 20 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a strictly monotone function. Then f is surjective.

☐ TRUE ☐ FALSE

Question 21 : The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \int_0^t |x| \, \mathrm{d}x$ is differentiable at $t = 0$.

☐ TRUE ☐ FALSE

Question 22 : If the series $\sum_{k=0}^{\infty} a_k \, (x-5)^k$ converges for $x = 2$, then it converges for $x = 6$.

☐ TRUE ☐ FALSE

Question 23 : Let $f : \mathbb{R} \to \mathbb{R}$ be a function with expansion of order 2 at $x_0 = 0$ given by $f(x) = a + bx + cx^2 + x^2\varepsilon(x)$ where a, b, c are real numbers. If f is differentiable at $x_0 = 0$, then $f'(0) = b$.

☐ TRUE ☐ FALSE

Question 24 : Let $z_1, z_2 \in \mathbb{C}$ be such that $\operatorname{Re}(z_1 \cdot z_2) = 0$. Then $\operatorname{Re}(z_1) \cdot \operatorname{Re}(z_2) = 0$.

☐ TRUE ☐ FALSE

Question 25 : Let $f\colon \,]0,1[\, \to \mathbb{R}$ be a continuous function. If $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x \to 1^-} f(x) = 0$, then f is bounded.

☐ TRUE ☐ FALSE

Question 26 : Let A and B be two nonempty bounded subsets of $\mathbb{R}$. If $\inf A > \sup B$, then $A \cap B$ is empty.

☐ TRUE ☐ FALSE

Question 27 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that the limit of the sequence $\big(f(\frac{1}{n})\big)_{n\geq 1}$ is $f(0)$. Then f is continuous at $x_0 = 0$.

☐ TRUE ☐ FALSE

Question 28 : Let $(a_n)_{n\geq 1}$ a sequence of strictly negative numbers. Then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if it converges.

☐ TRUE ☐ FALSE

Troisième partie, questions de type ouvert

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 29: *This question is worth 10 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 *Do not write here.*

(a) Let $a, b \in \mathbb{R}$ such that $a < b$. State the Mean Value Theorem for differentiable $f : [a, b] \to \mathbb{R}$.

(b) Give the definition of uniformly continuous for a function $f : \mathbb{R} \to \mathbb{R}$.

(c) Show carefully that if a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is bounded then f is uniformly continuous.

(d) Let $f : [1, 2] \to \mathbb{R}$ be continuous but not differentiable. What can be said about the uniform continuity of f?

(e) Give an example of a function that is differentiable but not C^1.

Question 30: *This question is worth 6 points.*

0 1 2 3 4 5 6 *Do not write here.*

(a) Define precisely the meaning of **a sequence** $(a_n)_{n\geq 0}$ **converges to** $l \in \mathbb{R}$.

(b) Show that if $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Indication: You may use without proof that for two sequences (a_n) and (b_n) such that their limits exist, $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.