ANALYSIS 1
(ENGLISH)
LECTURE 9

DEF LET (x_n) BE NON - CNV.

WE SAY THAT $\lim_{n \to \infty} x_n = \begin{matrix} +\infty \\ -\infty \end{matrix}$ IF

$\forall c \in \mathbb{R} \quad \exists n_c \in \mathbb{N}$ S.T

$$\forall n \geq n_c \implies x_n \begin{matrix} \geq \\ \leq \end{matrix} c$$

UNIQUENESS OF LIMITS HOLD

SQUEEZE AT ∞

PROP. (x_n), (y_n)

ASSUME $x_n \leq y_n \quad \forall n \geq n_0 \in \mathbb{N}$

SQUEEZE AT ∞

PROP. (x_n), (y_n)

$$x_n \leq y_n \quad \forall n \geq n_0$$

- IF $\lim_{n\to\infty} x_n = +\infty \Longrightarrow \lim_{n\to\infty} y_n = +\infty$

- IF $\lim_{n\to\infty} y_n = -\infty \Longrightarrow \lim_{n\to\infty} x_n = -\infty$

SQUEEZE AT ∞

PROP. (x_n), (y_n)

$$(*)\quad x_n \leq y_n \quad \forall n \geq n_0$$

① • $\lim_{n\to\infty} x_n = +\infty \Rightarrow \lim_{n\to\infty} y_n = +\infty$

• $\lim_{n\to\infty} y_n = -\infty \Rightarrow \lim_{n\to\infty} x_n = -\infty$

PROOF ① $\lim x_n = +\infty \overset{DEF}{\Longleftrightarrow} \forall C \in \mathbb{R}$

$\exists n_C \in \mathbb{N}$ s.t. $\forall n \geq n_C$ $\quad x_n \geq C$

$\overset{(*)}{\Longrightarrow} \forall n \geq \max\{n_0, n_C\} = n_C^+ \quad y_n \geq x_n \geq C$

EXAMPLE

$$x_n = \frac{n!}{c^n} \qquad c \in \mathbb{R}^*_+$$

$$x_n = \frac{n!}{c^n} = \underbrace{\frac{\overbrace{n(n-1)(n-2)\ldots 2\cdot 1}}{c\cdot c\cdot c \ldots c\cdot c}}_{n\text{-times}}$$

$$= \frac{n}{c}\,\frac{n-1}{c}\,\frac{n-2}{c}\ldots\frac{2}{c}\cdot\frac{1}{c}$$

IF $n \gg 0$: $\frac{n}{c} \to 1$, $\frac{n-1}{c} \to 1$, $\frac{n-2}{c} \to 1$

INTUITION: $\lim x_n = +\infty$?

ARCH. PROP. $\exists\, n_c \in \mathbb{N}$ S.T.

$$n_c > c \quad (\forall c \in \mathbb{R}_+)$$

c CHOSEN EARLIER

$$x_n = \frac{n!}{c^n} = \frac{n\,(n-1)(n-2)\cdots(n_c+1)\,n_c\,(n_c-1)!}{c\;\;c\;\;c\;\cdots\;c\;\;c\;\;c^{n_c-1}}$$

const.

$n \gg 0$

$n \geq n_c$

$$\underbrace{\frac{n}{c} \geq \frac{n_c}{c},\quad \frac{n-1}{c} \geq \frac{n_c}{c},\quad \frac{n-2}{c} \geq \frac{n_c}{c},\quad \cdots,\quad \frac{n_c+1}{c} \geq \frac{n_c}{c},\quad \frac{n_c}{c} \geq \frac{n_c}{c}}_{n-(n_c-1)\text{ - terms}} \qquad \frac{(n_c-1)!}{c^{n_c-1}}$$

$$\left(\frac{n_c}{c}\right)^{n+1-n_c} \cdot \frac{(n_c-1)!}{c^{n_c-1}} \leq$$

$$\Rightarrow \left(\frac{n_c}{c}\right)^{n} \cdot c' \longleftarrow c' = \frac{(n_c-1)!}{c^{n_c-1}} \left(\frac{n_c}{c}\right)^{1-n_c}$$

$$n_c > c$$

$$\forall n \geq n_c \qquad \underline{x_n \geq C' \cdot \left(\frac{n_c}{c}\right)^n}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{y_n}$$

SQUEEZE : If $\lim y_n = +\infty$

$\Longrightarrow \lim x_n = +\infty$

y_n = GEOM. SEQ. $q = \frac{n_c}{c} > 1$

$\Longrightarrow \lim y_n = +\infty$

MONOTONE SEQUENCES & LIMITS

(x_n) MONOTONE IF $\begin{cases} \text{INCREASING} \\ \forall n \quad x_n \geq x_{n-1} \\ \text{DECREASING} \\ \forall n \quad x_n \leq x_{n-1} \end{cases}$

MONOTONE SEQUENCES & LIMITS

(x_n) MONOTONE IF $\begin{cases} \text{INCREASING} \\ \text{DECREASING} \end{cases}$

EXAMPLE

- $x_n = n^3$ UNBOUNDED $\lim x_n = +\infty$
  INCREASING.

- $x_n = 1 + \left(\sqrt{n\sqrt{n}+16}\right)^{-1}$ BOUNDED $\lim x_n = 0$
  STRICTLY DECREASING
  CONV.

$$1 \geq \ldots \geq 0$$

$$\lim x_n = 1 + \lim_{n\to\infty} \frac{1}{\sqrt{n\sqrt{n}+16}} = 0, \qquad \sqrt{n\sqrt{n}+16} \to +\infty$$

$$\lim \sqrt{n\sqrt{n}+16} = +\infty$$

$$\Uparrow \quad \text{SQUEEZE}$$

$$\sqrt{n\sqrt{n}+16} > \sqrt{n} \qquad \forall n \in \mathbb{N}$$

$$\lim \sqrt{n} = +\infty$$

$$\lim \frac{1}{\sqrt{n\sqrt{n}+16}} = 0$$

$$\lim x_n = \lim 1 + \frac{1}{\sqrt{n\sqrt{n}+16}} = 1$$

MONOTONE SEQUENCES & LIMITS

(x_n) MONOTONE IF $\begin{cases} \text{INCREASING} \\ \text{DECREASING} \end{cases}$

Q: DOES $\lim_{n \to \infty} x_n$ ALWAYS EXIST?

[FINITE OR $\pm\infty$]

[POLL]

YES

ALWAYS!!

THEOREM (x_n) INCREASING.

① IF (x_n) BOUNDED FROM ABOVE $\Rightarrow \lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$

SET OF VALUES OF x_n

② IF (x_n) NOT BOUNDED $\Rightarrow \lim x_n = +\infty$

(x_n) DECREASING.

① IF (x_n) BOUNDED FROM BELOW $\Rightarrow \lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$

② IF (x_n) NOT BOUNDED $\Rightarrow \lim x_n = -\infty$

EXAMPLE

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

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$$x_n = \left(1 + \frac{1}{n}\right)^n$$

$$x_n = \left(1 + \frac{1}{n}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot \left(\frac{1}{n}\right)^i \cancel{1^{n-i}} =$$

BINOM. COEFF.

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!} = \frac{1}{i!} \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-i+1)}_{i \text{ TERMS}}$$

EXAMPLE

$$X_n = \left(1 + \frac{1}{n}\right)^n$$

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$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!} = \frac{1}{i!} \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-i+1)}_{i \text{ TERMS}}$$

$$\binom{n}{i} \frac{1}{\underset{\underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{i\text{-times}}}{\underset{\parallel}{n^i}}} = \frac{1}{i!} \quad \underset{\underset{1}{\underset{\parallel}{1}}}{\underset{\parallel}{\frac{n}{n}}} \quad \underset{\underset{1}{\underset{\wedge}{1-\frac{1}{n}}}}{\underset{\parallel}{\frac{n-1}{n}}} \quad \underset{\underset{1}{\underset{\wedge}{1-\frac{2}{n}}}}{\underset{\parallel}{\frac{n-2}{n}}} \quad \ldots \quad \underset{\underset{1}{\underset{\wedge}{1-\frac{i-1}{n}}}}{\underset{\parallel}{\frac{n-i+1}{n}}}$$

$$X_n \le \sum_{i=0}^{n} \frac{1}{i!}$$

X_n STRICTLY INCREAS.

$$X_n = \sum_{i=0}^{n} \frac{1}{i!}\, 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\left(\frac{i-1}{n}\right)\right)$$

$\wedge$

$$\binom{n+1}{i}\left(\frac{1}{n+1}\right)^{i} 1^{(n+1)-i}$$

$$X_{n+1} = \sum_{i=0}^{n+1} \frac{1}{i!}\, 1\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{(i-1)}{n+1}\right)$$

$$\left(1-\frac{1}{n}\right) \qquad 1-\frac{2}{n} \qquad 1-\frac{i-1}{n}$$

$$\wedge \qquad\qquad \wedge \qquad\qquad \wedge$$

$$\sum_{i=0}^{n} \frac{1}{i!}\, 1\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{i-1}{n+1}\right) + \left(\frac{1}{n+1}\right)^{n+1}$$

$$\vee$$

$$0$$

EXAMPLE $\quad x_n = \left(1 + \frac{1}{n}\right)^n$ STRICTLY INCREASING

x_n CONV. $\Longleftrightarrow$ (x_n) BOUNDED

$$x_n = \sum_{i=0}^{n} \frac{1}{i!} \, 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \left(\frac{i-1}{n}\right)\right)$$

$$\wedge\!\backslash$$

$$\sum_{i=0}^{n} \frac{1}{i!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{i!}$$

$$\boxed{\frac{1}{\begin{matrix} 1 & 2 & 3 & 4 & \cdots & (2k-1)2k \\ & 2 & & 2 & 2 & \end{matrix}}} = \frac{1}{(2k)!} \leq \frac{1}{2^k}$$

$$(2k)! \geq 2^k$$

$$\frac{1}{(2k+1)!} \geq 1/2^k$$

$$X_n = \left(1+\frac{1}{n}\right)^n \quad \text{STR. INCR.}$$

$$X_n \le 4 \implies \text{CONV.}$$

$$0 \le \sum_{i=0}^{n} \frac{1}{i!} \le 4 \implies X_n$$

$$\sum_{i=0}^{n} \frac{1}{i!} = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} \frac{1}{i!} + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \frac{1}{i!}$$

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{n} \frac{1}{i!} = \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{2j!} \le \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{2^j}$$

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \frac{1}{i!} = \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{(2j+1)!} \le \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{2^j}$$

$$\sum_{i=0}^{n} \frac{1}{i!} \le 2 \cdot \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{2^j}$$

$$\frac{-\left(\frac{1}{2}\right)^{[\frac{n}{2}]+1} + 1}{1-\frac{1}{2}} \le 2$$

$$e \doteq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$\shortparallel$

NEPERO'S

#

EXAMPLE $\quad x_n = \left(1 + \frac{1}{n}\right)^n$ STRICTLY INCREASING

BOUNDED $\leq 4 \quad \forall n$

$\Longrightarrow$

SUBSEQUENCES

EXAMPLE $x_n = (-1)^n$

$y_n \doteq x_{2n} = 1$ CONST.

$z_k \doteq x_{2k+1} = (-1)$ CONST.

DEF FOR A SEQUENCE (x_n), A SUBSEQUENCE IS ANOTHER SEQUENCE (y_k) DEFINED AS

VARIABLE FOR THE INDEX

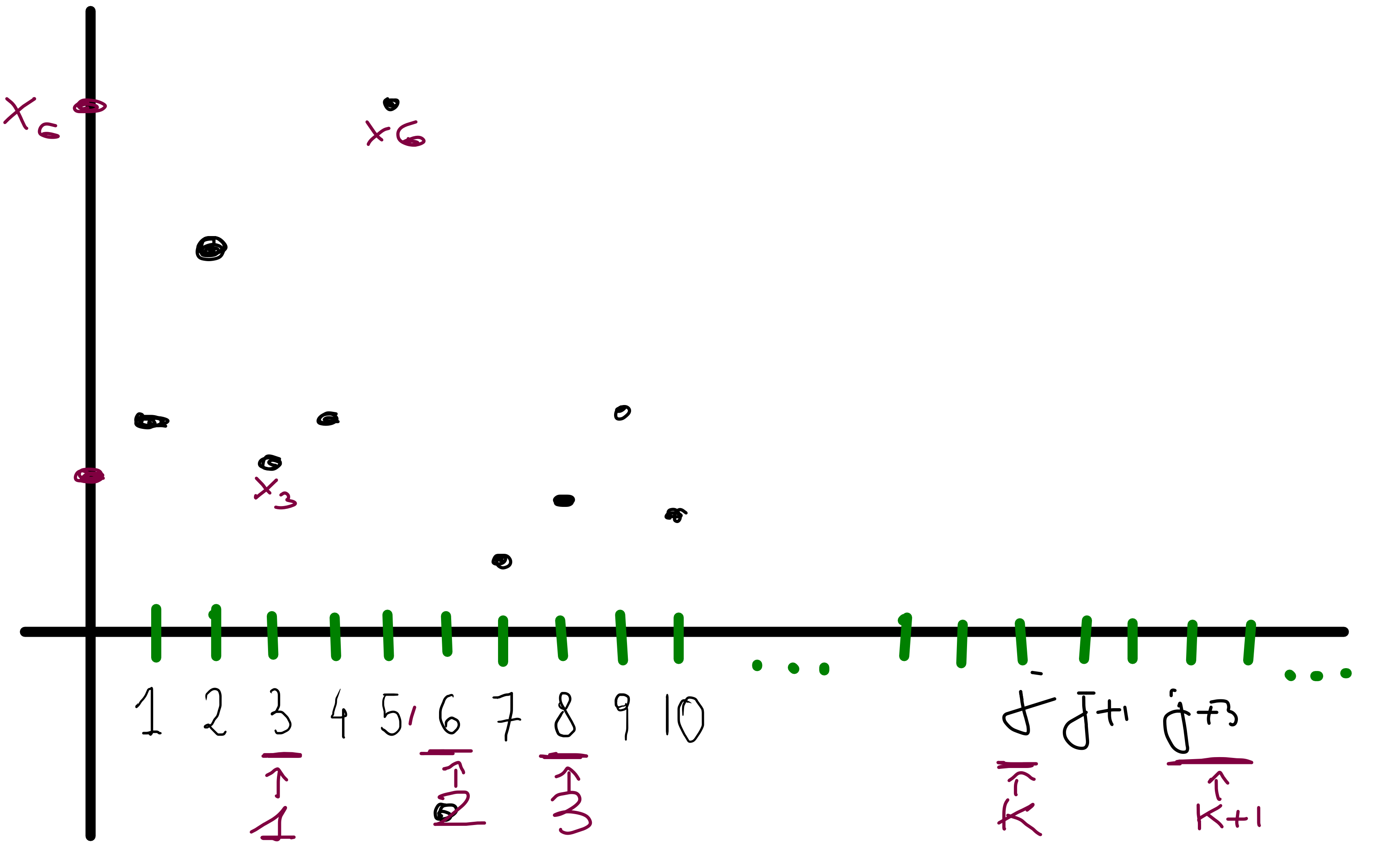
$$y_k \doteqdot x_{n_k}, \text{ WHERE}$$

$$f : \mathbb{N} \ni \underline{\underline{k}} \xmapsto[\text{INCREASING}]{\text{STRICTLY}} \underline{\underline{n_k}} \in \mathbb{N}$$

$f(k) \doteqdot$

$f(k+1) > f(k)$

$y_k \doteqdot x_{f(k)} = x_{n_k}$



$y_1 \doteq x_3 \quad y_2 \doteq x_6 \quad y_3 \doteq x_8 \quad y_K \doteq x_{j+1} \quad y_{K+1} \doteq x_{j+3}$

EXAMPLE: $n_k \doteq 2k$ [SUBSEQ. OF EVEN INDICES]

$x_n = (-1)^n \rightsquigarrow y_k = x_{n_k} = 1$

CONSTANT

$\{2, 4, 6, 8, 10, 12, \ldots, 2k\}$

$f : \mathbb{N} \longrightarrow \mathbb{N}$

$k \longmapsto 2k$

$y_k \doteq x_{2k} = 1$

EXAMPLE

- $n_k \doteq 2k$ [SUBSEQ. OF EVEN INDICES]

$$x_n = (-1)^n \longrightarrow y_k = x_{n_k} = 1$$

CONSTANT

- $n_k \doteq 2k+1$ [SUBSEQ. OF ODD INDICES]

$$x_n = \left(\frac{2}{5}\right)(-3)^n \longrightarrow y_k = x_{2k+1} = \left(\frac{2}{5}\right)(-3)^{2k+1}$$

$$= \left(\frac{-6}{5}\right) 9^k$$

$$\{1, 3, 5, 7, 9, \ldots, 2k+1, \ldots\}$$

$$\begin{array}{cccc} 1 & 3 & 5 & 7 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 0 & 1 & 2 & 3 \end{array}$$

$$f(k) = 2k+1$$

EXAMPLE

• $n_k \doteq 2k$ [SUBSEQ. OF EVEN INDICES]

$$x_n = (-1)^n \rightsquigarrow y_k = x_{n_k} = 1$$

CONSTANT

• $n_k \doteq 2k+1$ [SUBSEQ. OF ODD INDICES]

$$x_n \doteq \left(\frac{2}{5}\right)(-3)^n \longrightarrow y_k = x_{2k+1} = \left(-\frac{6}{5}\right) 9^k \xrightarrow{n\to\infty} -\infty$$

$f(k) = 5k$

• $n_k \doteq 5k$

$$x_n \doteq \left(1+\frac{5}{n}\right)^{2n} \rightsquigarrow y_k = x_{5k} = \left(1+\frac{5}{5k}\right)^{10k} = \left(\left(1+\frac{1}{k}\right)^k\right)^{10}$$

$$\lim y_k = e^{10}$$

EXAMPLE

$$x_n = 1 + (-1)^n \frac{1}{\sqrt{n}}$$

CONV. SEQ

$$\lim x_n = 1 + \lim (-1)^n \underbrace{\frac{1}{\sqrt{n}}}_{=0} = 1$$

$$n_k \doteq 2k \longrightarrow y_k = x_{2n} = 1 + \frac{1}{\sqrt{k}}$$

CONV. SEQ
OF lim = 1

$$n_k \doteq k^2 \longrightarrow y_k = 1 + (-1)^{k^2} \frac{1}{k}$$

$$y_k = x_{k^2}$$

$f(k) = k^2$

$$\lim y_k = \cancel{0}\ 1$$

EXAMPLE

$$x_n = 1 + (-1)^n \frac{1}{\sqrt{n}}$$

$$n_k \doteq 2k \longrightarrow y_k = x_{2n} = 1 + \frac{1}{\sqrt{2n}}$$ CONVERGENT

$$n_k \doteq k^2 \longrightarrow y_k = 1 + (-1)^{k^2} \frac{1}{k}$$ CONVERGENT

Q: IS IT ALWAYS THE CASE ?

PROP (x_n) $\lim_{n\to\infty} x_n = x \in \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

IF $y_k = x_{n_k}$ SUBSEQUENCE

$$\Rightarrow \lim_{k\to\infty} y_k = x$$

PROP (x_n) $\lim_{n\to\infty} x_n = x$

$y_k = x_{n_k}$ SUBSEQUENCE

$$\Rightarrow \lim_{k\to\infty} y_k = x$$

USEFUL! IF (x_n) CONVERGES WE CAN FIND THE LIMIT BY COMPUTING IT ALONG A SUBSEQUENCE

$$y_k = x_{n_k}$$

EXAMPLE $\cdot \quad n_k \doteq 5k$

$$x_n \doteq \left(1+\frac{5}{n}\right)^{2n} \rightsquigarrow y_k \doteq x_{5k} = \left(1+\frac{5}{5k}\right)^{10k} = \left(\left(1+\frac{1}{k}\right)^{k}\right)^{10}$$

$$y_n \longrightarrow e^{10}$$

MOREOVER: x_n STR INCR. $\Rightarrow$ x_n CONV.
x_n BOUNDED

$$\lim x_n = \sup \{ x_n \mid n \in \mathbb{N} \} = e^{10}$$

EXAMPLE

- $$x_n = \left(\frac{n+5}{n}\right)^{10n}$$

EXAMPLE

- $n_k \doteq 5k$

$$x_n \doteq \left(1+\frac{5}{n}\right)^{2n} \rightsquigarrow y_k = x_{5k} = \left(1+\frac{5}{5k}\right)^{10k}$$

$$y_k \longrightarrow e^{10}$$

PROP $\Longrightarrow$ IF x_n CONVERGES THEN $\lim_{n\to\infty} x_n = e^{10}$

EXAMPLE • $x_n = (-1)^n$ BOUNDED

NON-CONV.

$y_k = x_{2k} = 1 \quad \forall k \in \mathbb{N}$ CONV.

$z_k = x_{2k+1} = (-1) \quad \forall k \in \mathbb{N}$ CONV.

$\exists k \in \mathbb{N}$ s.t. $x_m = x_{m+k}$ $\forall m \in \mathbb{N}$

• $x_n = \sin(n)$ BOUNDED

Q: $\exists$ CONVERGING SUB'S ?

THEOREM (BOLZANO-WEIERSTRASS)

(x_n) BOUNDED

$\Longrightarrow \exists (y_k)$ SUBSEQUENCE $[y_k = x_{m_k}]$

WHICH IS CONVERGENT

EXAMPLE $x_n = (-1)^n$ BOUNDED DIVERGENT

$x_{2k} = 1 \quad \forall k \in \mathbb{N}$ CONST. [CONV.]

$x_{2k+1} = (-1) \quad \forall k \in \mathbb{N}$ CONST. [CONV.]

DIFFERENT SUBSEQUENCES MAY HAVE DIFFERENT LIMITS

EXAMPLE

$x_n \doteq \sin(n)$

(x_n) BOUNDED

B-W THM $\Longrightarrow$ $\exists$ a converging subsequence

THEOREM (BOLZANO-WEIERSTRASS)

(x_n) ~~BOUNDED~~

$\Longrightarrow \exists (y_k)$ SUBSEQUENCE $[y_k = x_{m_k}]$

WHICH IS CONVERGENT

FALSE!!!

THEOREM (BOLZANO-WEIERSTRASS)

(x_n) ~~BOUNDED~~

$\Longrightarrow \exists\, (y_k)$ SUBSEQUENCE $\left[y_k = x_{n_k} \right]$

WHICH IS CONVERGENT

FALSE!!!

EXAMPLE $x_n = n$

STR. INCR.
UNBOUNDED

THEOREM (BOLZANO-WEIERSTRASS)

(x_n) ~~BOUNDED~~

$\Longrightarrow \exists (y_k)$ SUBSEQUENCE $[y_k = x_{m_k}]$

WHICH IS CONVERGENT

FALSE!!!

EXAMPLE $x_n = n$ STRICTLY INCR.

ANY SUBSEQUENCE IS ALSO STRICTLY MONOTONE + UNBOUNDED - -

PROOF OF B-W

ASSUME THAT (x_n) ADMITS A MONOTONE SUBSEQUENCE:

$$y_k = x_{n_k}$$

PROOF OF B-W

ASSUME THAT (x_n) ADMITS A MONOTONE SUBSEQUENCE:

$$y_k = x_{n_k}$$

$\Longrightarrow$ y_k IS BOUNDED! $\Longrightarrow$ y_k CONVERGES

$$\{y_k \mid k \in \mathbb{N}\} \subseteq \{x_n \mid n \in \mathbb{N}\}$$

$y_k = x_{n_k}$

BOUNDED

PROOF OF B-W

ASSUME THAT (x_n) ADMITS A MONOTONE SUBSEQUENCE:

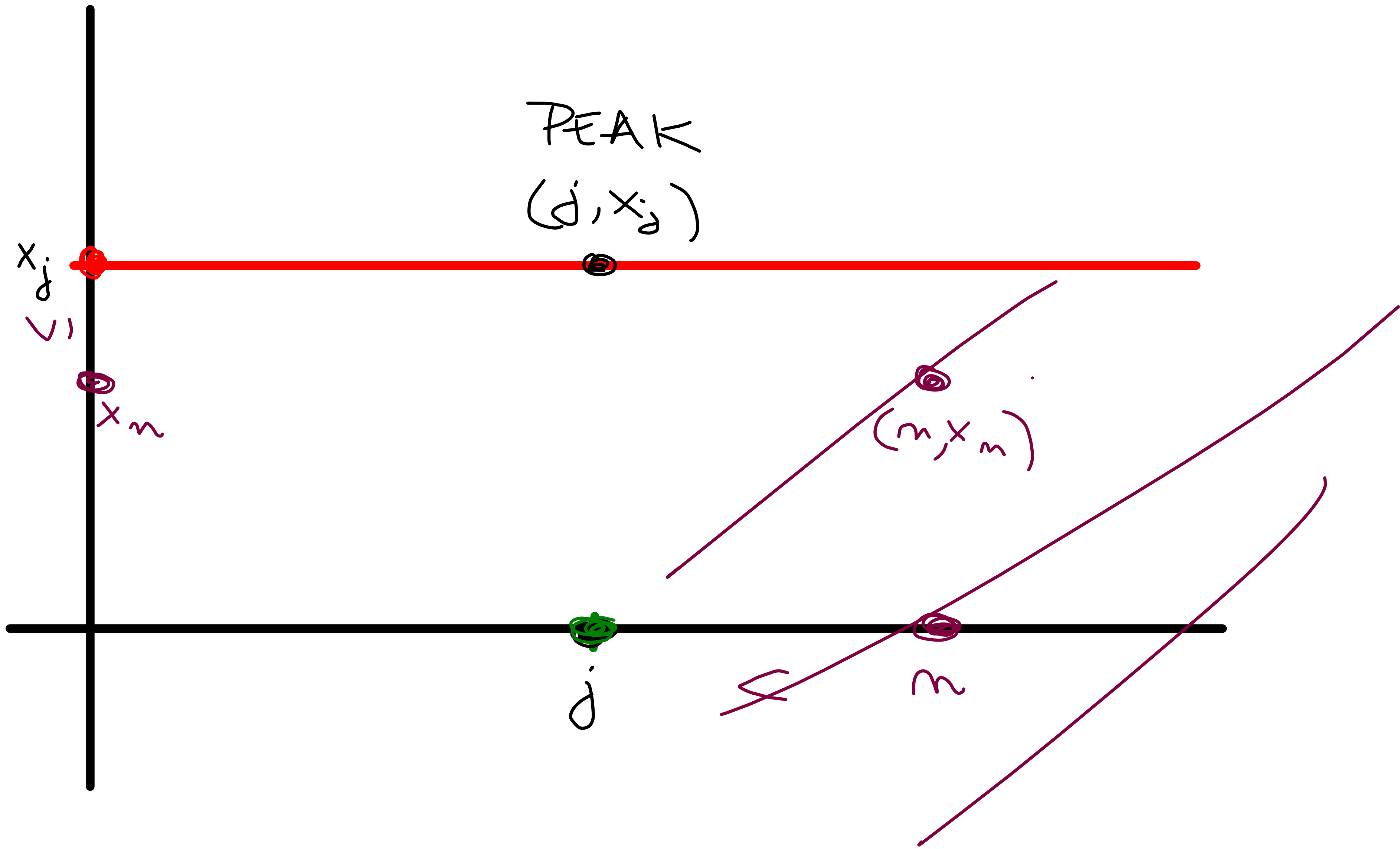
$$y_k = x_{n_k}$$

$\Longrightarrow$ y_k IS BOUNDED! $\Longrightarrow$ y_k CONVERGES

HOW DO WE CONSTRUCT A MONOTONE SUBSEQ.?

DEF AN ELEMENT x_j IS A PEAK IF

$$\forall n > j \quad x_n \leq x_j$$



PEAK
(j, x_j)
x_j
V1
x_n
(n, x_n)
j
n

PROOF OF B-W

ASSUME THAT (x_n) ADMITS A MONOTONE SUBSEQUENCE:

$$y_k = x_{n_k}$$

$\Rightarrow$ y_k IS BOUNDED! $\Rightarrow$ y_k CONVERGES

HOW DO WE CONSTRUCT A MONOTONE SUBSEQ.?

DEF AN ELEMENT x_j IS A PEAK IF

$$\forall n > j \quad x_n \leq x_j$$

WHAT ABOUT PEAKS :

∞ - MANY PEAKS OF x_n

FINITELY MANY ($\emptyset$)

$\exists K$ s.t. x_k IS THE LAST PEAK

$y_0 = x_k$

$y_1 = x_{k+1} \leftarrow$ NOT A PEAK $\Rightarrow \exists j > k+1$ s.t. $x_j > x_{k+1}$

$y_2 = x_j \qquad \exists j_1 > j$ s.t. $x_{j_1} > x_j$

$y_3 = x_{j_1}$

y_k = TAKE $x_{j_k} > x_{j_{k-1}}$

y_k MONOT. STR. INCR.

$y_1 = x_{n_1}$ 1ST PEAK DECR. SEQ

$y_2 = x_{n_2}$ 2nd PEAK $y_k \geq y_{k+1}$

$y_k = x_{n_k}$ kth PEAK

$y_{k+1} = x_{n_{k+1}}$ ($k+1$)st PEAK $\Longrightarrow$ $x_{n_k} \overset{\text{PEAK}}{\geq} x_{n_{k+1}}$

CAUCHY SEQUENCES

EXAMPLE

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

CAUCHY SEQUENCES

EXAMPLE

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

$$x_n \xrightarrow{n \to \infty} \ell$$

CAUCHY SEQUENCES

EXAMPLE $x_n = \left(1 + \frac{1}{n}\right)^n$

$$x_n \xrightarrow{n \to \infty} e$$

HOW DID WE PROVE IT ?

CAUCHY SEQUENCES

EXAMPLE $x_n = \left(1+\frac{1}{n}\right)^n$

$$x_n \xrightarrow{n\to\infty} e$$

HOW DID WE PROVE IT?

1. (x_n) IS STRICTLY INCREASING

CAUCHY SEQUENCES

EXAMPLE

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

$$x_n \xrightarrow{n \to \infty} e$$

HOW DID WE PROVE IT?

1. (x_n) IS STRICTLY INCREASING
2. (x_n) IS BOUNDED

CAUCHY SEQUENCES

EXAMPLE $\quad x_n = \left(1 + \frac{1}{n}\right)^n$

$$x_n \xrightarrow{n \to \infty} e$$

HOW DID WE PROVE IT ?

1. (x_n) IS STRICTLY INCREASING
2. (x_n) IS BOUNDED

$\overset{\text{THM}}{\Longrightarrow}$ LIMIT EXISTS

IN GENERAL: GIVEN (x_n)

IF WE CAN GUESS "$x_n \xrightarrow{n\to\infty} x$"

IN GENERAL: GIVEN (x_n)

IF WE CAN GUESS $x_n \xrightarrow{n\to\infty} x$

THEN WE CAN VERIFY IF

x SATISFIES THE DEF'N

OF LIMIT FOR (x_n)

IN GENERAL: GIVEN (x_n)

IF WE CAN GUESS $x_n \xrightarrow{n\to\infty} x$

SAME

THEN WE CAN VERIFY IF

x SATISFIES THE DEF'N

OF LIMIT FOR (x_n)

<u>EXAMPLE</u> $x_n = \frac{1}{n^2}$

IN GENERAL: GIVEN (x_n)

IF WE CAN GUESS $x_n \xrightarrow{n\to\infty} x$

SAME

THEN WE CAN VERIFY IF x SATISFIES THE DEF'N OF LIMIT FOR (x_n)

BUT WHAT IF WE CANNOT GUESS WHAT THE LIMIT MAY BE?

BUT WHAT IF WE CANNOT GUESS WHAT THE LIMIT MAY BE?

EXAMPLE

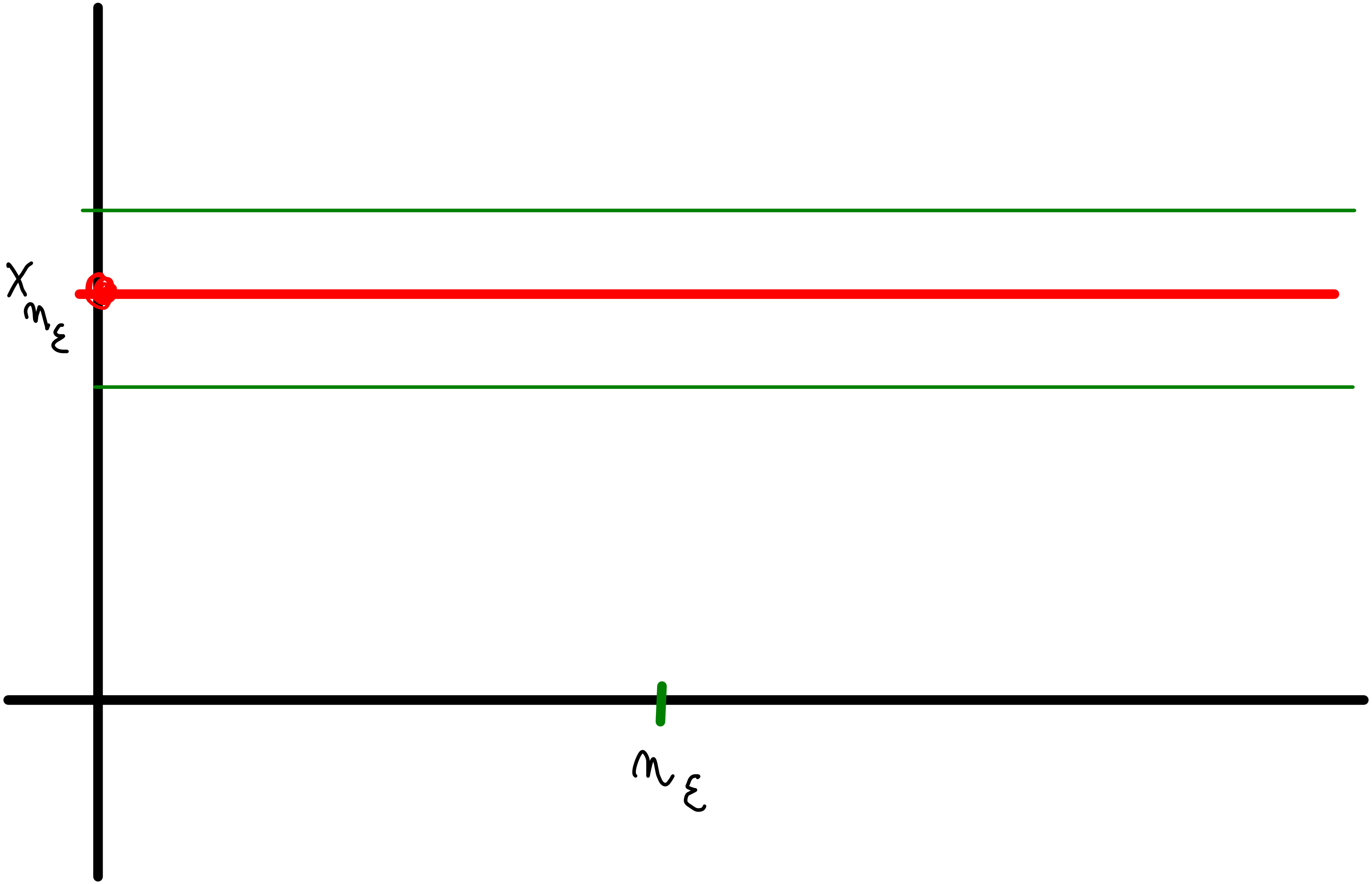
DEF (x_n) IS **A CAUCHY SEQUENCE**

IF $\forall \varepsilon > 0$ THERE IS $n_\varepsilon \in \mathbb{N}$

$\forall n, m \geq n_\varepsilon$ THEN

$\in \mathbb{N}$

$$|x_n - x_m| < \varepsilon$$



x_{n_ε}
n_ε