ANALYSIS 1
(ENGLISH)
LECTURE 8

TODAY:

1. LIMITS FOR RECURSIVE SEQUENCES
2. LIMITS $= \pm\infty$
3. MONOTONE SEQUENCES
4. SUBSEQUENCES

RECURSIVE SEQUENCES

IN GENERAL

$$\underline{x_n} \doteq \overset{\text{FIXED FORMULA}}{f}(x_{n-1}, x_{n-2}, \dots, x_{n-d})$$

d independent of n

$$+ \quad x_0 \doteq c_0, \; x_1 \doteq c_1, \dots, x_{d-1} \doteq c_{d-1}$$

EXAMPLE

- $x_n \doteq x_{n-1} + a$ $= an+b$ ARITHM. PROGR.
  $x_0 \doteq b$

- $x_n \doteq q\, x_{n-1}$, $x_0 = c$ GEOM PROGR.
  $= q^n \cdot c$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

INITIAL DATA

DOES THE LIMIT EXIST?

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

$\longrightarrow$ IF ≥ 1

DOES THE LIMIT EXIST?

- (y_n) BOUNDED?

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

DOES THE LIMIT EXIST?

- (y_n) BOUNDED?

$y_0 = 1$, $y_1 = 1 + \frac{1}{1} = 2$, $y_2 = 1 + \frac{1}{2} \doteq \frac{3}{2}$

$y_3 = 1 + \frac{2}{3} = \frac{5}{3}$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

- (y_n) BOUNDED?

$y_0 = 1$, $y_1 = 1 + \frac{1}{1} = 2$, $y_2 = 1 + \frac{1}{2} \doteq 2$

- $\underline{\underline{y_n \geq 1}}$: INDUCTION:
  - $y_0 \geq 1$ ✓ $n = 0$
  - IF $y_n \geq 1$
    $\Longrightarrow y_{n+1} = 1 + \frac{1}{y_n} \geq 1$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

NECESS CONDIT.

- (y_n) BOUNDED? YES

(*) • $y_n \geq 1$: INDUCTION:
- $y_0 \geq 1$
- IF $y_n \geq 1$ $\Longrightarrow$ $y_{n+1} = 1 + \dfrac{1}{y_n} \geq 1$

$\{y_n \mid n \in \mathbb{N}\} \subset [1,2]$

• $y_n \leq 2$: INDUCTION:
- $y_0 \leq 2$ $n = 0$
- IF $y_n \leq 2 \Longrightarrow y_{n+1} = 1 + \dfrac{1}{y_n} \leq 2$

≤ 1

$0 <$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} & (*) \\ y_0 = 1 \end{cases}$$

NECESS CONDIT.

- (y_n) BOUNDED? YES

SUPPOSE THE LIMIT EXISTS:

$$y \doteq \lim_{n\to\infty} y_n \implies y = 1 + \frac{1}{y}$$

$$y = \lim_{n\to\infty} y_n \overset{(*)}{=} \lim_{n\to\infty} 1 + \frac{1}{y_{n-1}}$$

$$= 1 + \lim_{n\to\infty} \frac{1}{y_{n-1}} = 1 + \frac{1}{y}$$

$$\lim 1 + \frac{1}{y_{n-1}}$$

$$\| \longleftarrow \text{assuming } \frac{1}{y_{n-1}} \text{ is converging}$$

$$1 + \lim \frac{1}{y_{n-1}}$$

$$\| \longleftrightarrow y_{n-1} \xrightarrow{n\to\infty} l \neq 0$$

$$1 + \frac{1}{\lim y_{n-1}} = 1 + \frac{1}{y}$$

$$z_n$$

$$\|$$

$$1 \leq y \overset{\leq 2}{=} \lim_{1 \leq} y_n^{\leq 2} = \lim y_{n-1}$$

$$z_n \geq s_n \qquad \lim z_n \geq \lim s_n$$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

NECESS CONDIT.

- (y_n) BOUNDED? YES

SUPPOSE THE LIMIT EXISTS: $0 < y \doteq \lim_{n\to\infty} y_n$

$$y = \lim_{n\to\infty} 1 + \frac{1}{y_{n-1}} = 1 + \frac{1}{y}$$

$\Longrightarrow$ y SOLVES EQUATION $x - 1/x - 1 = 0$

$$x - \frac{1}{x} - 1 = 0$$

$$\Updownarrow \quad x \neq 0$$

$$x^2 - 1 - x = 0 \quad (*)$$

SOL'S

$$x = \begin{cases} \dfrac{1+\sqrt{1+4}}{2} \\[2ex] \cancel{\dfrac{1-\sqrt{1+4}}{2} = \dfrac{1-\sqrt{5}}{2}} \end{cases}$$

$0 >$ $\quad$ Δ $\quad$ $\wedge$

FOR US y IS A SOL'N TO $(*)$

$$1 \leq y \leq 2 \implies y > 0 \implies y = \frac{1+\sqrt{5}}{2}$$

$$x - \frac{1}{x} - 1 = 0$$

$$\Updownarrow$$

$$x^2 - 1 - x = 0$$

SOL'S $\quad x = \begin{cases} \dfrac{1 + \sqrt{1+4}}{2} \\[2ex] \dfrac{1 - \sqrt{1+4}}{2} \end{cases}$

y IS A SOL & $1 \leq y \leq 2$

$$x - \frac{1}{x} - 1 = 0$$

$$\Updownarrow$$

$$x^2 - 1 - x = 0$$

SOL'S $\quad x = \begin{cases} \dfrac{1+\sqrt{1+4}}{2} \\[2ex] \dfrac{1-\sqrt{1+4}}{2} < 0 \end{cases}$

y IS A SOL & $1 \leq y \leq 2$

$$x - \frac{1}{x} - 1 = 0$$

$$\Updownarrow$$

$$x^2 - 1 - x = 0$$

SOL'S $\quad x = \begin{cases} \dfrac{1+\sqrt{5}}{2} = y \\[2ex] \cancel{\dfrac{1-\sqrt{1+4}}{2}} < 0 \end{cases}$

y IS A SOL & $1 \leq y \leq 2$

$$y = \frac{1 + \sqrt{5}}{2}$$

$$y = \frac{1 + \sqrt{5}}{2}$$

IS IT REALLY THE LIMIT?

$$y = \frac{1+\sqrt{5}}{2}$$

IS IT REALLY THE LIMIT?

NEED TO PROVE:

$$y = \frac{1+\sqrt{5}}{2}$$ IS IT REALLY THE LIMIT?

NEED TO PROVE:

$$\forall \varepsilon > 0 \,,\ \exists\, n_\varepsilon \in \mathbb{N} \text{ S.T.}$$

$$\forall n \geq n_\varepsilon \implies |y_n - y| < \varepsilon$$

$y = \dfrac{1+\sqrt{5}}{2}$ IS IT REALLY THE LIMIT?

NEED TO PROVE:

$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}$ S.T. $\forall n \geq n_\varepsilon \Rightarrow |y_n - y| < \varepsilon$

$y = 1 + \dfrac{1}{y}$

$|y_n - y| = \left|1 + \dfrac{1}{y_{n-1}} - \left(1 + \dfrac{1}{y}\right)\right| = \left|\dfrac{1}{y_{n-1}} - \dfrac{1}{y}\right| =$

$= \dfrac{|y_{n-1} - y|}{\underbrace{y_{n-1}}_{1 \leq} \cdot y} \leq \dfrac{|y_{n-1} - y|}{((1+\sqrt{5})/2) > 1}$

$$|y_n - y| \le \frac{1}{y}|y_{n-1} - y| \le \frac{1}{y^2}|y_{n-2} - y|$$

$$\underset{\frac{1}{y}|y_{n-2} - y|}{\wedge\!|} \qquad \underset{\frac{1}{y}|y_{n-3} - y|}{\wedge\!|}$$

$$\le \frac{1}{y^3}|y_{n-3} - y|$$

$$\left|1 - \frac{1+\sqrt{5}}{2}\right| \doteq C$$

$$|y_n - y| \le \frac{1}{y^n}|y_0 - y|$$

Fix $\varepsilon > 0$

$$|y_n - y| \le \frac{C}{y^n}$$

Choose n_ε s.t.

$$\frac{C}{y^{n_\varepsilon}} < \varepsilon \iff \frac{C}{\varepsilon} < y^{n_\varepsilon}$$

const

$$\forall n \ge n_\varepsilon \quad |y_n - y| \le \frac{C}{y^n} =$$

$\forall n \geq n_\varepsilon \quad |y_n - y| \leq \dfrac{c}{y^n} =$

$$= \frac{c}{y^{n_\varepsilon + (n - n_\varepsilon)}} = \frac{c}{y^{n_\varepsilon}} \cdot \frac{1}{y^{(n-n_\varepsilon)}}$$

$n - n_\varepsilon \geq 0$, $\quad y^{(n-n_\varepsilon)} \geq 1$

$$\leq \frac{c}{y^{n_\varepsilon}} < \varepsilon$$

y_n conv. $\Rightarrow$

$\lim y_n = y$

IN GENERAL:

RECIPE:

Q: DOES (x_n) CONVERGE?
IF SO, $\lim_{n\to\infty} x_n$?

$$\begin{cases} x_n = F(x_{n-1}, \dots, x_{n-j}) \\ x_0 = C_0, \dots, x_{j-1} = C_{j-1} \end{cases}$$

IN GENERAL: RECIPE:

$$\begin{cases} x_n = F(x_{n-1}, \dots, x_{n-j}) \\ x_0 = C_0, \dots, x_{j-1} = C_{j-1} \end{cases}$$

$x_n = 1 + \frac{1}{x_{n-1}}$

$\downarrow$

$y = 1 + \frac{1}{y}$

$\Downarrow y \neq 0$

$y^2 - 1 - y = 0$

1. GUESS VALUES OF THE LIMIT BY SOLVING $y = F(y, y, \dots, y)$
2. EXCLUDE ALL VALUES BUT 1
3. SHOW CONVERGENCE TO THAT VALUE [SQUEEZE, MONOTONE, ...]

EXAMPLE

$$\begin{cases} x_n = x_{n-1}^2 + 1 \\ x_0 = c \end{cases}$$

$x_1 = 1 + c^2 \geq 1$

$x_2 = 1 + x_1^2 \geq 2$

1 IF $y = \lim x_n$, THEN $x_3 = 1 + x_2^2 \geq 5$

$y = y^2 + 1 \implies y$ is A SOL'N OF $T^2 - T + 1 = 0$

$$\text{SOL'S} = \frac{1 \pm \sqrt{1-4}}{2}$$

$-3 < 0$

$T^2 - T + 1$ HAS NO REAL SOL'S

EXAMPLE

$$x_n = \frac{1}{2}\left(x_{n-1} + x_{n-2}\right)$$

EXAMPLE

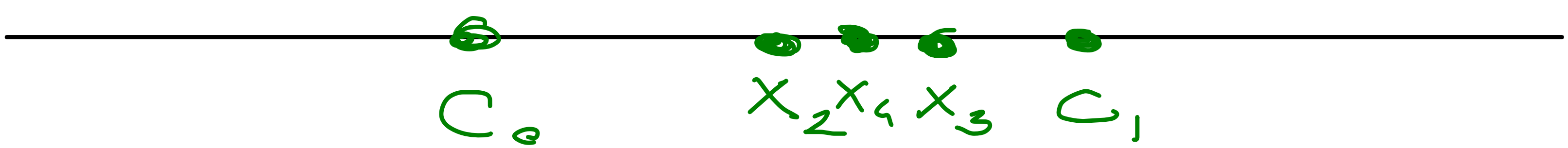
$$x_n = \frac{1}{2}\left(x_{n-1} + x_{n-2}\right)$$

① $$x = \frac{1}{2}\left(x + x\right) \iff x = x$$

$$\Updownarrow$$

$$0 = 0$$

$$\begin{cases} x_n = \frac{1}{2}\left(x_{n-1} + x_{n-2}\right) \\ x_0 = c_0 \\ x_1 = c_1 \end{cases}$$



EXAMPLE

$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$$

1) $x = \frac{1}{2}(x + x) = x$

ISSUE: IT DEPENDS ON INITIAL VALUES

$x_0 = 0 = x_1 \implies$

EXAMPLE

$$x_n = \frac{1}{2}\left(x_{n-1} + x_{n-2}\right)$$

1 $\quad x = \frac{1}{2}\left(x + x\right) = x$

ISSUE: IT DEPENDS ON INITIAL VALUES

$x_0 = 0 = x_1 \implies x_n = 0 \quad \forall n$

$x_0 = 1 = x_1 \implies x_2 = 1 =$

EXAMPLE

$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$$

1. $x = \frac{1}{2}(x + x) = x$

ISSUE: IT DEPENDS ON INITIAL VALUES

$x_0 = 0 = x_1 \implies x_n = 0 \quad \forall n$

$x_0 = 1 = x_1 \implies x_2 = 1 = x_3 = x_n \quad \forall n$

$x_0 = 1, \; x_1 = 2 \implies x_2 = 3/2, \; x_3 =$

DIVERGING TO $\pm\infty$

EXAMPLE

- $x_n = (-1)^n$

  BOUNDED BUT NOT CONV.

- $x_n = n^3$

  UNBOUNDED STRICTLY INCR $x_n > x_{n-1}$

- $x_n = (-1)^n n^2$

  UNBOUNDED

DEF LET (x_n)

WE SAY THAT $\lim_{n\to\infty} x_n = +\infty$ IF

$\forall C \in \mathbb{R} \quad \exists\, \underline{\underline{n_C}} \in \mathbb{N}$ S.T

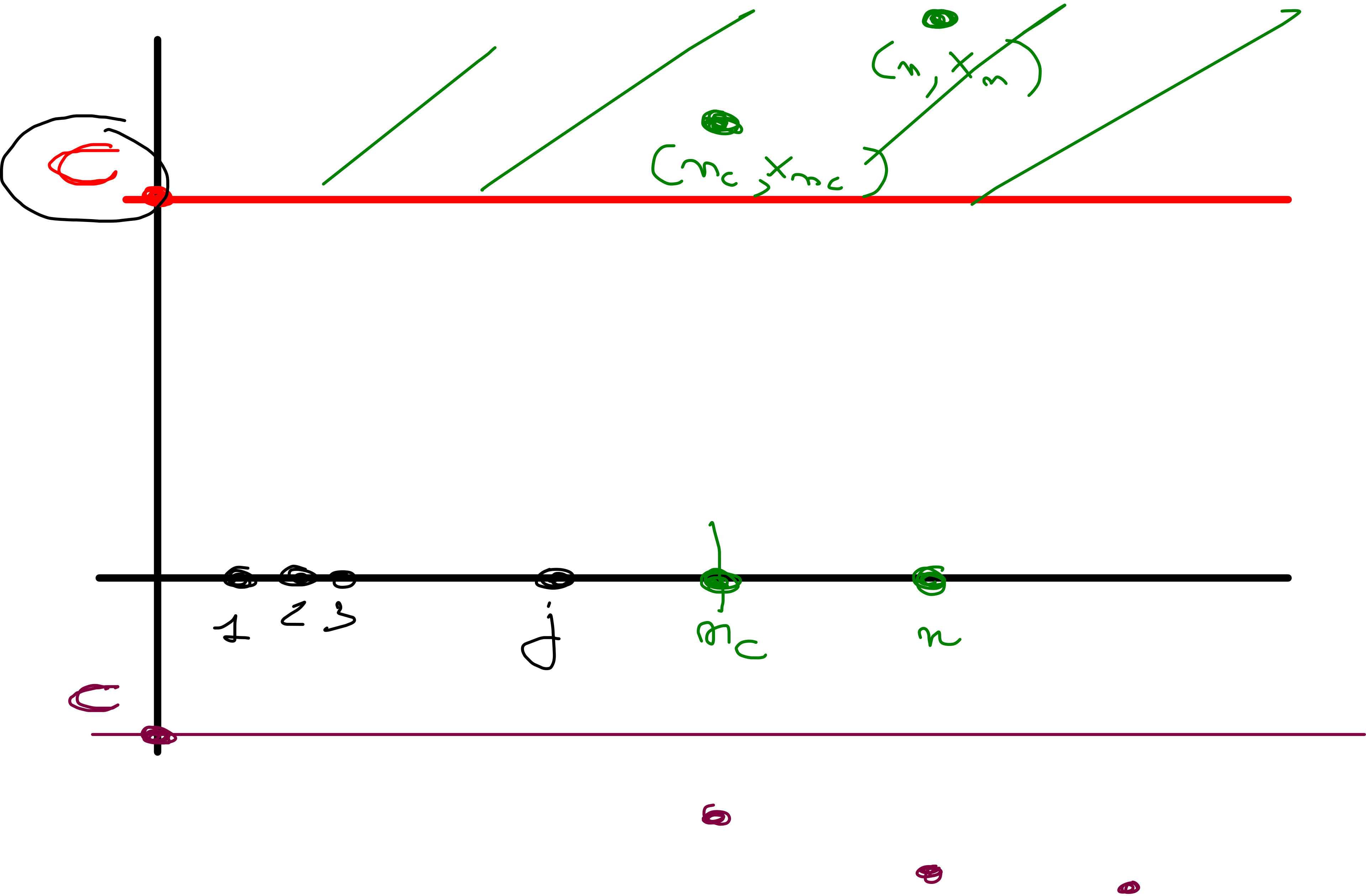
$\forall n \geq n_C \implies x_n \geq C$

DEF LET (x_n)

WE SAY THAT $\lim_{n\to\infty} x_n = +\infty$ / $-\infty$ IF

$\forall c \in \mathbb{R} \quad \exists n_c \in \mathbb{N}$ S.T

$\forall n \geq n_c \implies x_n \geq c$ / $x_n \leq c$



C
(m_c, x_{m_c})
(n, x_n)
1
2
3
j
m_c
n
C

EXAMPLE

- $x_n = n$ ARCH. PROP.

$$\lim x_n = +\infty$$

FIX $c \in \mathbb{R}$

- $c \leq 0 \Rightarrow x_n \geq c \quad \forall n > 0$
- $c > 0 \xRightarrow[\text{PROP}]{\text{ARCH.}}$

$\Longrightarrow \exists\, n_c \in \mathbb{N}$ s.t. $n_c > c$

$$n_c = \left[\frac{1}{c}\right] + 1$$

$\Longrightarrow \forall n \geq n_c, \quad \underset{x_n}{\underset{\parallel}{n}} \geq \underset{x_{n_c}}{\underset{\parallel}{n_c}} > c$

EXAMPLE

- $x_n = n$ $\qquad x_n \xrightarrow{n \to \infty} +\infty$

ARCH. PROPERTY

- $x_n = (-1)^n n^4$ IS [POLL]

1. ~~CONVERGENT~~ $\qquad x_n$ UNBOUNDED
2. $\lim x_n = +\infty$
3. $\lim x_n = -\infty$
4. NONE OF THE ABOVE

IF $\lim x_n = +\infty$

FIX $c \in \mathbb{R}$ WE NEED TO SHOW:

$\forall n \gg 0, \; x_n \geq c$

IF c is > 0 IMPOSSIBLE!

$x_n = (-1)^n n^5$ $x_{odd} < 0$

$x_{even} > 0$

EXAMPLE

• $x_n = a q^n$ GEOM. PROGR.

$a = 0 \Longrightarrow x_n = 0$

$a \neq 0$: – $|q| < 1 \Longrightarrow x_n \xrightarrow{n\to\infty} 0$

$q = 1 \Longrightarrow x_n = a$

$q = -1 \Longrightarrow x_n = (-1)^n a$

$|q| > 1$: $q > 1 \Longrightarrow \lim x_n = +\infty$

Fix $c \in \mathbb{R} \Longrightarrow$ NEED TO SHOW

$q^n > \frac{c}{a} \quad \forall n \geq n_c \in \mathbb{N}$

$\Downarrow$

$1 + n(q-1)$

$q < -1 \qquad x_n = aq^n$

x_{even} have the same sign as a

x_{odd} have the opposite sign as a

$\lim x_n$ doesn't exist at all

IN GENERAL: (x_n)

- BOUNDED
  - (x_n) CONV. & $\lim x_n = \bar{x} \in \mathbb{R}$
  - (x_n) NOT CONV.
    $[x_n = (-1)^n]$

- UNBOUNDED
  - $\lim x_n = +\infty$
  - $\lim x_n = -\infty$
  - x_n has no limit
    $[x_n = (-1)^n n^4]$

ALGEBRA & LIMIT = ∞

PROP $(x_n), (y_n)$

$\lim_{n \to \infty} x_n = +\infty$, (y_n) BOUNDED FROM BELOW

ALGEBRA & LIMIT $= \infty$

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = +\infty$, (y_n) BOUNDED FROM BELOW

THEN:

ALGEBRA & LIMIT $= \infty$

PROP (x_n), (y_n) $\quad (-1)^n$

$\lim_{n\to\infty} x_n = \underline{\underline{+\infty}}$, (y_n) BOUNDED FROM BELOW

THEN:

1. $\lim_{n\to\infty}(x_n + y_n) = +\infty$

ALGEBRA & LIMIT $= \infty$

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = +\infty$, (y_n) BOUNDED FROM BELOW

THEN:

1. $\lim_{n\to\infty} (x_n + y_n) = +\infty$

2. $\exists\, n_0$ S.T. $\forall n \geq n_0$, $y_n \geq A > 0$

ALGEBRA & LIMIT = ∞

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = +\infty$, (y_n) BOUNDED FROM BELOW

THEN:

① $\lim_{n\to\infty} (x_n + y_n) = +\infty$

② $\exists\, n_0$ S.T. $\forall n \geq n_0$, $y_n \geq A > 0$

THEN $\lim_{n\to\infty} x_n \cdot y_n = +\infty$

ALGEBRA & LIMIT = ∞

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = +\infty$, (y_n) BOUNDED FROM BELOW

THEN:

1. $\lim_{n\to\infty} (x_n + y_n) = +\infty$
2. $\exists\, n_0$ S.T. $\forall n \geq n_0$, $y_n \geq A > 0$

   THEN $\lim_{n\to\infty} x_n\, y_n = +\infty$
3. (y_n) BOUNDED $\Rightarrow \lim_{n\to\infty} y_n/x_n = 0$

ALGEBRA & LIMIT = ∞

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = +\infty$, $\boxed{(y_n) \text{ BOUNDED FROM BELOW}}$

IMPORTANT!!

THEN:

① $\lim_{n\to\infty} (x_n + y_n) = \infty$

② $\exists\, n_0$ S.T. $\forall n \geq n_0$, $y_n \geq A > 0$

THEN $\lim_{n\to\infty} x_n \cdot y_n = +\infty$

③ (y_n) BOUNDED $\Rightarrow \lim_{n\to\infty} y_n / x_n = 0$

ALGEBRA & LIMIT = ∞

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = +\infty$, $\boxed{(y_n) \text{ BOUNDED FROM BELOW}}$

IMPORTANT!!

DO NOT REMOVE!!

THEN:

1. $\underline{\lim_{n\to\infty}(x_n + y_n) = +\infty}$

2. $\exists\, n_0$ S.T. $\forall n \geq n_0$, $y_n \geq A > 0$

   THEN $\lim_{n\to\infty} x_n \cdot y_n = +\infty$

3. (y_n) BOUNDED $\Rightarrow \lim_{n\to\infty} y_n/x_n = 0$

EXAMPLE • $x_n = n$, $y_n = -n$ ← NOT BOUNDED FROM BELOW

$$x_n + y_n = 0 \quad \forall n \qquad \text{CONV.}$$

• $x_n = n$, $y_n = -n + (-1)^n$

$$x_n + y_n = (-1)^n$$ NOT CONV. + BOUNDED

• $x_n = n$, $y_n = -n^2$

$$x_n + y_n = n - n^2 \xrightarrow{n \to \infty} -\infty$$

ALGEBRA & LIMIT $= \infty$

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = -\infty$, (y_n) BOUNDED FROM ABOVE

THEN:

1. $\lim_{n\to\infty} (x_n + y_n) = -\infty$

2. $\exists\, n_0$ S.T. $\forall n \geq n_0$, $y_n \leq A < 0$ $\quad y_n > A > 0$

   THEN $\lim_{n\to\infty} x_n y_n = +\infty$ $[-\infty]$.

3. (y_n) BOUNDED $\Rightarrow \lim_{n\to\infty} y_n / x_n = 0$

ALGEBRA & LIMIT = ∞

[SEE P. 29-30 BOOK]

PROP (x_n), (y_n)

$\lim_{n \to \infty} x_n = -\infty$, (y_n) BOUNDED FROM ABOVE

THEN:

① $\lim_{n \to \infty} (x_n + y_n) = -\infty$

② $\exists\, n_0$ S.T. $\forall n \geq n_0$, $\boxed{y_n \leq A < 0}$ (*)

THEN $\lim_{n \to \infty} x_n \, y_n = +\infty$

③ (y_n) BOUNDED $\Rightarrow \lim_{n \to \infty} y_n / x_n = 0$

$x_n = n \quad , \; y_n = \frac{-1}{n} > 0$

$\lim y_n = 0$

$\Longrightarrow \nexists A > 0$ s.t.

$y_n \le -A \quad \forall n \gg 0$

$x_n \cdot y_n = -1$

$\nexists A > 0$ s.t.

$y_n \le -A$ OR $y_n \ge A$ $\forall n \gg 0$

$\lim x_n \cdot y_n = -1$ $\quad \lim y_n = 0$

OTHER EX. : $x_n = n \; , \; y_n = \frac{(-1)^n}{n}$

$x_n \cdot y_n = (-1)^n$

EXAMPLE

- $z_n = 2^n + \dfrac{\cos(e^n+3)}{\sqrt{n^2+9-6n}}$

$2^n = x_n$, $\dfrac{\cos(e^n+3)}{\sqrt{n^2+9-6n}} = y_n$

$\lim x_n = +\infty$

IF y_n B.F.B. $\lim x_n + y_n = +\infty$

$y_n = \dfrac{\cos(e^n+3)}{\sqrt{n^2+9+6n}}$, $\cos(e^n+3) \geq -1$

$\sqrt{n^2+9+6n} = \sqrt{(n+3)^2} = |n+3| \geq 3$

$y_n \geq -\dfrac{1}{3} \quad \forall n \geq 0 \quad$ B.F.B

- $\lim x_n = 0 \implies \lim \dfrac{1}{x_n} = ?$

(1) $\dfrac{1}{x_n}$ MAKES SENSE ONLY IF $x_n \neq 0$

[OR AT LEAST $x_n \neq 0 \quad \forall n \geq n_0$]

• If $x_n > 0 \quad \forall n \geq n_0$ $\left[\lim x_n = 0\right]$

$$\Longrightarrow \lim \frac{1}{x_n} = +\infty$$

• If $x_n < 0 \quad \forall n \geq n_0$

$$\Longrightarrow \lim \frac{1}{x_n} = -\infty$$

$\forall \varepsilon > 0 \ \exists n_\varepsilon$ s.t.

$\forall n \geq n_\varepsilon \quad x_n \leq \varepsilon \iff \frac{1}{x_n} \geq \frac{1}{\varepsilon}$

$\frac{1}{x_n} \geq \underset{\infty}{c} \iff x_n \leq \frac{1}{c} = \varepsilon$