ANALYSIS 1

(ENGLISH)

LECTURE 7

TODAY

1. SQUEEZE THM
2. QUOTIENT CRITERIA
3. RECURSIVE SEQ'S
4. LIMIT TO ∞

PROPERTIES OF LIMITS:

$(x_n), (z_n)$ SEQUENCES

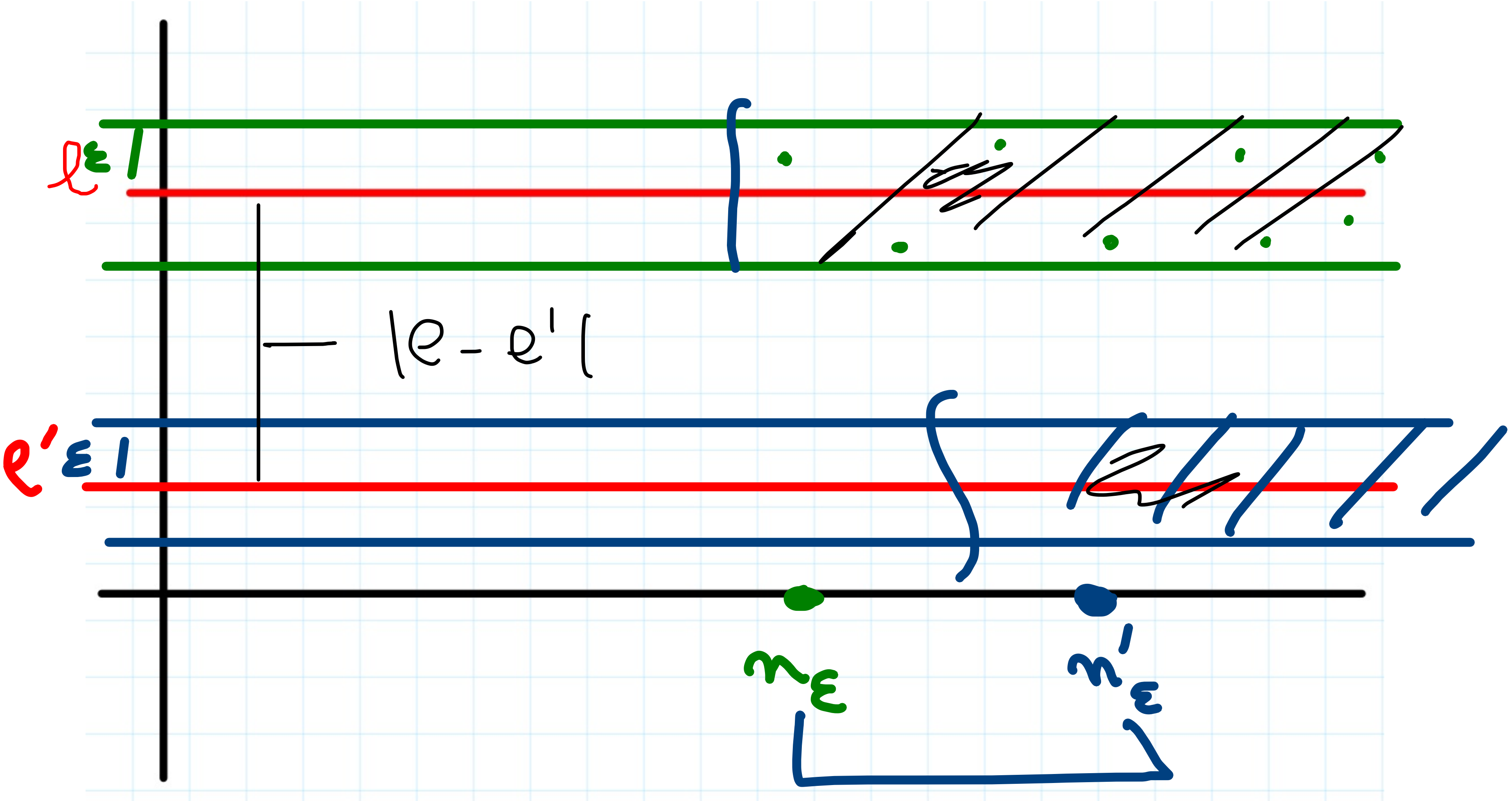
- UNIQUENESS: if $\lim_{n\to\infty} x_n = \ell \in \mathbb{R}$

THEN IT IS UNIQUE

- ORDER: if $\lim_{x_n\to\infty} = \ell$, $\lim_{z_n\to\infty} = \ell'$ $\ell, \ell' \in \mathbb{R}$

if $\underline{\underline{x_n \leq z_n}}$ $\forall n \geq n_0 \in \mathbb{N}$

THEN $\ell \underline{\leq} \ell'$



THE ℓ BAND ARE DISJOINT FOR $\varepsilon < |\ell - \ell'|/2$

LIMITS & ALGEBRA

PROP (x_n), (z_n)

$$\lim_{n\to\infty} x_n = \overline{x}, \quad \lim_{n\to\infty} z_n = \overline{z}.$$

THEN:

① $$\lim_{n\to\infty} (x_n + z_n) = \overline{x} + \overline{z}$$

② $$\lim_{n\to\infty} (x_n \cdot z_n) = \overline{x} \cdot \overline{z}$$

③ $$\lim_{n\to\infty} \frac{x_n}{z_n} = \frac{\overline{x}}{\overline{z}}$$ [ASSUMING $\overline{z} \neq 0$]

PROOF $\left[\lim_{n\to\infty}(x_n+z_n)=\lim_{n\to\infty}x_n+\lim_{n\to\infty}z_n\right]$

PROOF $\left[\lim_{n\to\infty}(x_n+z_n)=\lim_{n\to\infty}x_n+\lim_{n\to\infty}z_n\right]$

$$y \doteq \lim_{n\to\infty}x_n+\lim_{n\to\infty}z_n \quad , \; y\in\mathbb{R}$$

LIMIT OF A SEQUENCE

DEF SEQUENCE $(x_n + z_n)$ CONVERGES TO $y \in \mathbb{R}$ IF FOR EVERY $\varepsilon > 0$ THERE EXISTS $n_\varepsilon \in \mathbb{N}$ SUCH THAT

$$\forall n \geq n_\varepsilon \implies |(x_n + z_n) - y| < \varepsilon$$

PROOF $\left[\lim_{n\to\infty} \underbrace{(x_n + z_n)}_{= y_n} = \lim_{n\to\infty} x_n + \lim_{n\to\infty} z_n \right]$

$$y \doteq \lim_{n\to\infty} x_n + \lim_{n\to\infty} z_n \quad , \quad y \in \mathbb{R} \qquad y = x + z$$

$$\lim x_n \doteq x \qquad \lim z_n = z$$

Fix $\varepsilon > 0$.

$\exists\, n_{\varepsilon/2}, n'_{\varepsilon/2} \in \mathbb{N}$ S.T.

$$|x_n - x| \leq \varepsilon/2 \quad \forall n \geq n_{\varepsilon/2}$$

$$|z_n - z| \leq \varepsilon/2 \quad \forall n \geq n'_{\varepsilon/2}$$

$$\Big| \underbrace{(x_n + z_n)}_{= y_n} - \underbrace{y}_{= x+z} \Big| = \big| \underbrace{(x_n - x)}_{a} + \underbrace{(z_n - z)}_{b} \big| \leq \qquad |a + b| \leq |a| + |b|$$

$$\cancel{x}\,\|(x_n - x) + (z_n - z)\| \leq$$

$$\overset{\Delta}{\leq}$$

$$|x_n - x| + |z_n - z| \leq \varepsilon \quad \forall n \geq n'_{\varepsilon/2}$$

$$\underbrace{|x_n - x|}_{\leq \varepsilon/2 \ \forall n \geq n_{\varepsilon/2}} \qquad \underbrace{|z_n - z|}_{\leq \varepsilon/2 \ \forall n \geq n'_{\varepsilon/2}}$$

$$n''_{\varepsilon/2} \doteq \max\left(n_{\varepsilon/2},\ n'_{\varepsilon/2}\right)$$

SQUEEZE THM

$(x_n), (y_n), (z_n)$

ASSUME $x_n \le y_n \le z_n \quad \forall n \ge n_0 \in \mathbb{N}$

SQUEEZE THM

$(x_n), (y_n), (z_n)$

ASSUME $\underline{\underline{x_n}} \le y_n \le \underline{\underline{z_n}} \quad \forall n \ge n_0$

& $\displaystyle\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} z_n$

THEN $\displaystyle\lim_{n\to\infty} y_n = x.$

EXAMPLE GEOM PROGRESSION

$$x_n = a q^n$$

LAST TIME: FOR $|q| > 1$ (x_n) UNBOUND.

$\Longrightarrow$ NON - CONV.

Q : $|q| \leq 1$?

$|q| = 1$

$q = 1$ $x_n = a(1)^n = a$ CONST. $\Longrightarrow$ CONV.

$q = (-1)$ $x_n = (-1)^n = a$ — $a = 0$ CONST. $\Rightarrow$

— $a \neq 0 \Longrightarrow \begin{cases} a & n \text{ even} \\ -a & n \text{ odd} \end{cases}$

$|q| < 1$ $|a q^n| \leq |a| |q|^n$ $\lim x_n = 0$

$0 < q < 1$ CONVERG. $-1 < q < 0$ CONV.

$$|q| < 1 \iff \left|\frac{1}{q}\right| > 1$$

$$\left|\frac{1}{q}\right|^n \geq 1 + n\left(\left|\frac{1}{q}\right| - 1\right) \qquad \forall n \geq 0$$

$$\geq 0 \qquad\qquad n\left(\left|\frac{1}{q}\right| - 1\right) \qquad \bigcap \; \mathbb{N}$$

$$\underset{x_n}{0} \leq \underset{y_n}{|q|^n} \leq \underset{z_n}{\frac{1}{n} \cdot \underset{C}{\frac{1}{\left|\frac{1}{q}\right| - 1}}} \quad \forall n \in \mathbb{N}$$

SQUEEZE: $\lim x_n = 0$

$\lim z_n = \lim \frac{1}{n} \cdot C = 0$ $\implies \lim |q|^n = 0$

EXAMPLE

GEOM PROGRESSION

$$x_n = a q^n$$

LAST CASE: $a \neq 0 \quad |q| < 1$

$$x_n = 10^6 \cdot \left(\frac{1}{3}\right)^n$$ IS

1. DIVERGENT
2. CONVERGENT TO 0
3. CONVERGENT TO 10^6
4. NONE OF THE ABOVE

EXAMPLE

GEOM PROGRESSION

$$x_n = a q^n$$

$$x_n = 10^6 \cdot \left(\frac{1}{3}\right)^n \xrightarrow{n\to\infty} 0$$

BERN. INEQ

$$3^n \geqslant 1 + 2n$$

$$\Rightarrow \underset{x_n}{0} \leq \underset{y_n}{\frac{1}{3^n}} \leq \frac{1}{1+2n} \leq \underset{z_n}{\frac{1}{n}}$$

$$x_n \xrightarrow{n\to\infty} 0 \xleftarrow{n\to\infty} z_n \Rightarrow \frac{1}{3^n} \xrightarrow{n\to\infty} 0$$

EXAMPLE

$$x_n = \frac{2^n}{n!}$$

EXAMPLE

$$x_n = \frac{2^n}{n!} = \frac{2 \cdot 2 \cdot \ldots \cdot 2}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} \leftarrow n \text{ times}$$

EXAMPLE

$$x_n = \frac{2^n}{n!} = \frac{2 \cdot 2 \cdot \ldots \cdot 2}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}$$

$$= \underbrace{\underset{\substack{\| \\ 2}}{\frac{2}{1}} \cdot \underset{\substack{\| \\ 1}}{\frac{2}{2}}} \cdot \underset{\substack{\wedge \\ 1}}{\frac{2}{3}} \cdot \underset{\substack{\wedge \\ 1}}{\frac{2}{4}} \cdot \underset{\substack{\wedge \\ 1}}{\frac{2}{5}} \cdot \ldots \cdot \underset{\substack{\wedge \\ 1}}{\frac{2}{n-1}} \cdot \underset{\substack{\wedge \\ 1}}{\frac{2}{n}}$$

INTUITION: $\lim_{n \to \infty} x_n = 0$

$$x_n = \underset{\substack{\| \\ \frac{2^{n-1}}{(n-1)!}}}{x_{n-1}} \cdot \frac{2}{n}$$

EXAMPLE

$$x_n = \frac{2^n}{n!} = \frac{2\cdot 2\cdot \ldots \cdot 2}{1\cdot 2\cdot 3\cdot \ldots \cdot n}$$

INTUITION: $\lim_{n\to\infty} x_n = 0$

SQUEEZE!!

$$0 \leqslant \frac{2^n}{n!} = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\underbrace{\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{2}{3}}_{n-2 \text{ TIMES}} \qquad \forall n \in \mathbb{N}$$

$$\frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{2}{5}\cdots\frac{2}{n}, \qquad \frac{2}{3} \geq \frac{2}{4},\ \frac{2}{3} \geq \frac{2}{5},\ \ldots,\ \frac{2}{3} \geq \frac{2}{n}$$

$x_n \qquad y_n \qquad z_n$

$$z_n = \frac{2}{1}\cdot\frac{2}{2}\cdot\left(\frac{2}{3}\right)^{n-2} = \frac{9}{2}\cdot\frac{2^n}{3^{n}}$$

$$= \frac{2^{n-2}}{3^{n-2}} \qquad 9\cdot 2\, 3^{n-2}$$

$$\frac{9}{2}\cdot\left(\frac{2}{3}\right)^n \Longrightarrow \lim z_n = 0$$

$$\frac{2}{1}\cdot\frac{2}{2} \qquad \frac{2^{n-2}}{3^{n-2}} = \frac{2^{n}}{2\cdot 3^{n-2}}$$

$$= \frac{9}{3^{2}}\cdot\frac{2^{n}}{2\cdot 3^{n-2}} =$$

$$\frac{9\cdot 2^{n}}{2\cdot 3^{n}}$$

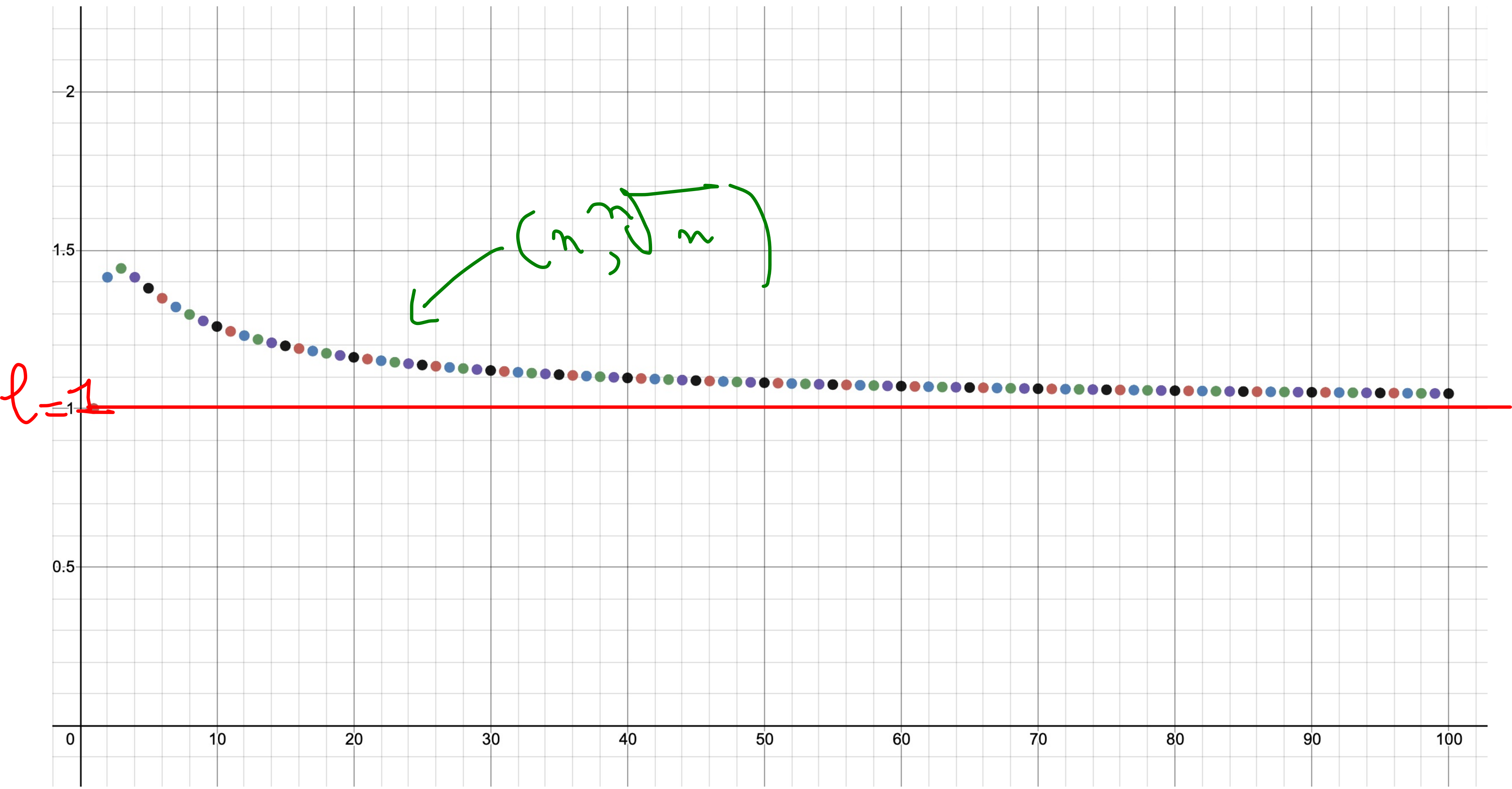
EXAMPLE

$$x_n \doteq \sqrt[n]{n} = (n)^{1/n}$$

EXAMPLE

$$x_n \doteq \sqrt[n]{n} = (n)^{1/n}$$

$$1 < \sqrt[n]{n} \leqslant \; ?$$



INTUITION $\quad \lim_{n \to \infty} \sqrt[n]{n} = 1$

EXAMPLE

$$x_n \doteq \sqrt[n]{n} = (n)^{1/n}$$

$$1 \leq \sqrt[n]{n} \leq ?$$

NAIVE TRY: $\sqrt[n]{n} \leq 1 + \frac{1}{n} \quad \forall n \geq n_0$

$$\Updownarrow$$

NOT POSSIBLE

$$n \leq \underbrace{\left(1 + \frac{1}{n}\right)^n}_{= z_n} \quad \not\forall n \geq n_0$$

z_n CONVERGING $\longrightarrow e$

WANT TO PROVE:

$$1 \le \sqrt[n]{n} \le 1 + \frac{1}{\sqrt{n}}$$

$x_n \qquad y_n \qquad z_n$

SQUEEZE: $\lim z_n = 1$

$\lim y_n = 1$

$$y_n \le z_n \iff n \le \left(1 + \frac{1}{\sqrt{n}}\right)^n$$

$\forall n \ge n_0$

EXAMPLE

$$x_n \doteq \sqrt[n]{n} = (n)^{1/n}$$

$\exists\, n_0 \in \mathbb{N}$ s.t. $\forall\, n \geq n_0$

$$\left(1+\frac{1}{\sqrt{n}}\right)^n \geq n$$

$$\Downarrow$$

$$1 < \sqrt[n]{n} \leq 1+\frac{1}{\sqrt{n}}$$

EXAMPLE

$$x_n \doteq \sqrt[n]{n} = (n)^{1/n}$$

$$\exists\, n_0 \in \mathbb{N} \quad \text{s.t.} \quad \forall\, n \geq n_0$$

$$\left(1+\frac{1}{\sqrt{n}}\right)^n \geq n$$

$$\Downarrow$$

$$1 < \sqrt[n]{n} \leq 1+\frac{1}{\sqrt{n}}$$

SQUEEZE:

$$n < \left(1+\frac{1}{\sqrt{n}}\right)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} \left(\frac{1}{\sqrt{n}}\right)^i$$

$$= 1 + \binom{n}{1}\frac{1}{\sqrt{n}} + \binom{n}{2}\cdot\frac{1}{n}$$

OTHER TERMS

$$+ \cdots \quad \geq 0$$

$$\geq 1 + \sqrt{n} + \frac{\cancel{n}(n-1)}{2}\cdot\frac{1}{\cancel{n}} = \frac{n-1}{2} < n$$

$$+\binom{n}{3}\cdot\left(\frac{1}{\sqrt{n}}\right)^3 = \frac{\cancel{n}(n-1)(n-2)}{6}\cdot\frac{1}{\cancel{n}\sqrt{n}}$$

WANT TO PROVE:

$$\frac{(n-1)(n-2)}{6} \cdot \frac{1}{\sqrt{n}} \geq n \qquad \forall n \geq n_0$$

WANT TO PROVE:

$$(*) \quad \frac{(n-1)(n-2)}{6} \cdot \frac{1}{\sqrt{n}} \geq n$$

CLAIM $\exists\, n_0$ S.T. $\forall\, n \geq n_0$ $(*)$ TRUE!

WANT TO PROVE:

$$(*)\quad \frac{(n-1)(n-2)}{6} \cdot \frac{1}{\sqrt{n}} \geqslant n$$

CLAIM $\exists\, n_0$ S.T. $\forall n \geqslant n_0$ $(*)$ TRUE!

PROOF EQUIVALENT TO SHOWING:

$$\forall n \geqslant n_0, \quad \frac{6\, n \cdot \sqrt{n}}{(n-1)\cdot(n-2)} \leqslant 1$$

WANT TO PROVE:

$$(*) \quad \frac{(n-1)(n-2)}{6}\cdot\frac{1}{\sqrt{n}} \geq n$$

CLAIM $\exists\, n_0$ S.T. $\forall\, n \geq n_0$ $(*)$ TRUE!

PROOF EQUIVALENT TO SHOWING:

$$\forall\, n \geq n_0, \quad \frac{6n\sqrt{n}}{(n-1)\cdot(n-2)} \leq 1$$

$$\lim_{n\to\infty} s_n = \frac{6n\sqrt{n}}{(n-1)\cdot(n-2)} = 0 \quad \text{[EXERCISE]}$$

Fix $\varepsilon = 1$ $\exists\, n_1$ s.t. $\forall\, n \geq n_1$ $|s_n| \leq 1$

PROOF OF SQUEEZE THM

$$x_n \leqslant y_n \leqslant z_n$$

$$x_n \xrightarrow{n\to\infty} x \xleftarrow{n\to\infty} z_n$$

WANT TO SHOW: $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ S.T.

$$n \geqslant n_\varepsilon \implies |y_n - x| < \varepsilon$$

FIX $\varepsilon > 0$

PROOF OF SQUEEZE THM $x_n \le y_n \le z_n$

WANT TO SHOW: $\forall \varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}$ s.t.

$$n \ge n_\varepsilon \implies |y_n - x| < \varepsilon$$

Fix $\varepsilon > 0 \implies \exists\, n_\varepsilon, n'_\varepsilon \in \mathbb{N}$ s.t.

$$n \ge n'_\varepsilon \implies |x_n - x| < \varepsilon$$

$$n \ge n''_\varepsilon \implies |z_n - x| < \varepsilon$$

$$x_n \leq y_n \leq z_n \qquad x_n \xrightarrow{n\to\infty} x \xleftarrow{n\to\infty} z_n$$

Fix $\varepsilon > 0 \implies \exists\, n_\varepsilon, n'_\varepsilon \in \mathbb{N}$ s.t.

$$n \geq n_\varepsilon \implies |x_n - x| < \varepsilon$$

$$n \geq n'_\varepsilon \implies |z_n - x| < \varepsilon$$

$$\Downarrow$$

$$< x_n \leq y_n \leq z_n <$$

EXAMPLE

$$x_n = \frac{\sin(n)}{n^2}$$

EXAMPLE

$$x_n = \frac{\sin(n)}{n^2} \xrightarrow[n\to\infty]{} 0 \quad \leftarrow \text{BOUNDED}$$

$$0 \leq |x_n| = \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2} \xrightarrow{n\to\infty} 0$$

$$s_n \leq |x|_n \leq r_n$$

SQUEEZE: $\lim s_n = 0$, $\lim r_n = 0$ $\not\Longrightarrow$ $\lim |x|_n = 0$ $\overset{(*)}{=}$ $\lim x_n = 0$

$$(*) \quad \lim_{n\to\infty} |x_n| = 0 \Longleftrightarrow \lim x_n = 0$$

new sequence

IN GENERAL:

PROP (x_n), (y_n)

$$\lim_{n \to \infty} x_n = 0$$

IN GENERAL:

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = 0$ & (y_n) BOUNDED

IN GENERAL:

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = 0$ & (y_n) BOUNDED

$$\Longrightarrow \lim_{n\to\infty} x_n \cdot y_n = 0$$

$$\underline{0} \leq |x_n \cdot y_n| = |x_n| \cdot \underbrace{|y_n|}_{\leq L} \leq \underbrace{L \cdot |x_n|}_{\to 0}$$

IN GENERAL:

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = 0$ & (y_n) BOUNDED

$\Longrightarrow \lim_{n\to\infty} x_n \cdot y_n = 0$

IN GENERAL:

PROP $(x_n), (y_n)$

$\lim_{n\to\infty} x_n = 0$ & (y_n) BOUNDED

$\Longrightarrow \lim_{n\to\infty} x_n \cdot y_n = 0$

REALLY NECESSARY!!

IN GENERAL:

PROP (x_n), (y_n)

$\lim_{n\to\infty} x_n = 0$ & (y_n) BOUNDED

$\Longrightarrow \lim_{n\to\infty} x_n \cdot y_n = 0$

REALLY NECESSARY!!

COUNTEREXAMPLE: $x_n = 1$, $y_n = (-1)^n$

$$x_n \cdot y_n = (-1)^n$$

COROLLARY

(x_n)

ASSUME :

$$\lim_{n\to\infty} \underbrace{\frac{|x_n|}{|x_{n-1}|}}_{z_n} = q$$

COROLLARY

(x_n)

WE ASSUME THIS LIMIT $\exists$

ASSUME : $\lim_{n\to\infty} \frac{|x_n|}{|x_{n-1}|} = q$

COROLLARY (x_n)

WE ASSUME THIS LIMIT $\exists$

ASSUME: $\lim_{n\to\infty} \frac{|x_n|}{|x_{n-1}|} = q \geq$

$z_n \geq 0$

$\downarrow$

$\ell \geq 0$

- IF $q < 1 \implies x_n \xrightarrow{n\to\infty} 0$
- IF $q > 1 \implies (x_n)$ UNBOUNDED [NOT CONV.]

"QUOTIENT CRITERION"

EXAMPLE

$$\begin{cases} x_n = \left(C + (-1)^n \frac{1}{n^2+1}\right) \cdot x_{n-1} \\ x_0 = 1 \end{cases}$$

$$x_1 = \left(C + (-1)^{1} \frac{1}{1^2+1}\right) x_0$$

$$\lim \frac{|x_n|}{|x_{n-1}|} = \lim_{n\to\infty} \left| C + (-1)^n \cdot \frac{1}{n^2+1} \right|$$

$$= |C|, \qquad \frac{1}{n^2+1} \to 0$$

QUOT. CRIT. :

$|C| < 1 \implies x_n \to 0$

$|C| > 1 \implies x_n$ unbounded

$C = 1$?

WHAT HAPPENS WHEN $|X_n / X_{n-1}| \xrightarrow{n\to\infty} 1$

- $X_n = 2n$ UNBOUNDED

  DOES NOT CONV.

$$\lim \frac{|X_n|}{|X_{n-1}|} = \lim \frac{\cancel{2} n}{\cancel{2}(n-1)} = 1$$

- $X_n = \frac{2}{n} \xrightarrow{n\to\infty} 0$

$$\frac{|X_n|}{|X_{n-1}|} = \frac{2}{n} \Big/ \frac{2}{n-1} = \frac{n-1}{n} \xrightarrow{n\to\infty} 1$$

- $X_n = (-1)^n$ $\qquad |X_n| = 1$

BOUNDED + NON-CONV. $\lim \frac{|X_n|}{|X_{n-1}|} = 1$

RECURSIVE SEQUENCES

$$x_n \doteq x_{n-1} + x_{n-2}$$ [FIBONACCI]

$x_0 = 0$

$x_1 = 1$

$x_2 = 1$

$x_3 = 2$

$x_4 = 3$

$x_5 = 5$

$x_6 = 8$

$x_7 = 13$

21

$x_0 = 0 = x_1$

$x_2 = 0$

$x_3 = 0$

CONST. 0

RECURSIVE SEQUENCES

$$x_n \doteq x_{n-1} + x_{n-2}$$ [FIBONACCI]

$x_0 = 1$

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

$x_4 = 5$

$x_5 = 8 \quad \ldots$

IN GENERAL

$$x_n \doteq f(x_{n-1}, x_{n-2}, \ldots, x_{n-d})$$

RECURSIVE SEQUENCES

$$x_n \doteq x_{n-1} + x_{n-2}$$ [FIBONACCI]

$f(x,y) = x+y$

$x_0 = 1$

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

$x_4 = 5$

$x_5 = 8 \quad \ldots$

IN GENERAL

$$x_n \doteq f(x_{n-1}, x_{n-2}, \ldots, x_{n-d})$$

FIXED FORMULA

DOES NOT DEPEND FROM n

$+$

$x_0 \doteq c_0$

$\vdots$

$x_{d-1} \doteq c_{d-1}$

RECURSIVE SEQUENCES

$$x_n \doteq x_{n-1} + x_{n-2}$$ [FIBONACCI]

$x_0 = 1$

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

$x_4 = 5$

$x_5 = 8 \quad \ldots$

IN GENERAL

$$x_n \doteq f(x_{n-1}, x_{n-2}, \ldots, x_{n-d})$$

$$+ \quad x_0 \doteq c_0$$

$\vdots$

$$x_{d-1} \doteq c_{d-1}$$

CONSTANTS $\in \mathbb{R}$

RECURSIVE SEQUENCES

IN GENERAL

$$x_n \doteq f(x_{n-1}, x_{n-2}, \dots, x_{n-d})$$

$+ \quad x_0 \doteq c_0, \; x_1 \doteq c_1, \dots, \; x_{d-1} \doteq c_{d-1}$

EXAMPLE

RECURSIVE SEQUENCES

IN GENERAL $x_n \doteq f(x_{n-1}, x_{n-2}, \dots, x_{n-d})$

$+ \quad x_0 \doteq c_0, \quad x_1 \doteq c_1, \dots, \quad x_{d-1} \doteq c_{d-1}$

EXAMPLE

$$x_n \doteq x_{n-1} + a$$
$$x_0 = b$$

ARITHM. PROGR. OF FACTOR a

$x_0 = b, \quad x_1 = b + a, \quad x_2 = b + 2a$

$x_3 = b + 3a, \quad \boxed{x_n = b + n \cdot a}$

$$\begin{cases} x_n \doteq x_{n-1} + a \\ x_0 \doteq b \end{cases}$$

$$\text{"}x_n = b + n \cdot a\text{"} = P(n)$$

BY IND:

(1) INIT. ST.: $P(0)$ holds ✓

$$x_0 = b = b + 0 \cdot a$$

(2) IND. ST. $P(n-1)$ SAT. $\Rightarrow P(n)$ SAT.

$$x_{n-1} \text{"}= b + (n-1)a$$

$$x_n = x_{n-1} + a = b + (n-1)a + a = b + n \cdot a$$

RECURSIVE SEQUENCES

IN GENERAL $x_n \doteq f(x_{n-1}, x_{n-2}, \dots, x_{n-d})$

$+ \quad x_0 \doteq c_0, \quad x_1 \doteq c_1, \dots, \quad x_{d-1} \doteq c_{d-1}$

EXAMPLE

- $x_n \doteq x_{n-1} + a$, $x_0 \doteq b$ ARITHM. PROGR.

- $x_n \doteq q\, x_{n-1}$, $x_0 = c$

RECURSIVE SEQUENCES

IN GENERAL

$$x_n \doteq f(x_{n-1}, x_{n-2}, \dots, x_{n-j})$$

$$+ \quad x_0 \doteq c_0, \quad x_1 \doteq c_1, \dots, x_{j-1} \doteq c_{j-1}$$

EXAMPLE

- $x_n \doteq x_{n-1} + a$, $x_0 \doteq b$ ARITHM. PROGR.

- $x_n \doteq q\, x_{n-1}$, $x_0 = c$ GEOM. PROGR.

$$x_n = q^n \cdot c$$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

DOES THE LIMIT EXIST?

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

DOES THE LIMIT EXIST?

- (y_n) BOUNDED?

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

DOES THE LIMIT EXIST?

- (y_n) BOUNDED?

$y_0 = 1$,

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

DOES THE LIMIT EXIST?

- (y_n) BOUNDED?

$y_0 = 1$, $y_1 = 1 + \dfrac{1}{1} =$

LIMITS OF RECURSIVE SEQUENCES

$$\begin{cases} y_n = 1 + \dfrac{1}{y_{n-1}} \\ y_0 = 1 \end{cases}$$

DOES THE LIMIT EXIST?

- (y_n) BOUNDED?

$y_0 = 1$, $y_1 = 1 + \frac{1}{1} = 2$, $y_2 = 1 + \frac{1}{2} = 2$

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$\Longrightarrow$

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- $y_n \leq 2$: INDUCTION:
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$$y \doteq \lim_{n \to \infty} y_n$$

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$\Longrightarrow$ y SOLVES EQUATION $x - 1/x - 1 = 0$

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SOL'S $\quad x = \begin{cases} \dfrac{1 + \sqrt{1+4}}{2} \\[2ex] \dfrac{1 - \sqrt{1+4}}{2} \end{cases}$

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SOL'S $\quad x = \begin{cases} \dfrac{1+\sqrt{5}}{2} = y \\[2ex] \cancel{\dfrac{1-\sqrt{1+4}}{2}} < 0 \end{cases}$

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$1 \leq y_{n-1}$

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$$= \left|\frac{y - y_{n-1}}{y_{n-1} \cdot y}\right| \leq \frac{1}{y} |y_{n-1} - y|$$

$$1 \leq y_{n-1}$$

$$\frac{1}{y} < 1$$

$$|y_n - y| \leq \frac{1}{y}|y_{n-1} - y|$$

$$\frac{1}{y}|y_{n-2} - y|$$

IN GENERAL: RECIPE:

$$\begin{cases} x_n = F(x_{n-1}, \ldots, x_{n-j}) \\ x_0 = c_0, \ldots, x_{j-1} = c_{j-1} \end{cases}$$

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1. GUESS VALUES OF THE LIMIT BY SOLVING $y = F(y, y, \dots, y)$
2. EXCLUDE ALL VALUES BUT 1
3. SHOW CONVERGENCE TO THAT VALUE [SQUEEZE, MONOTONE, ...]